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Paths, tableaux and q-characters of quantum affine algebras: the C_n case

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Abstract

For the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ with $\mathfrak{g}$ of classical type, let $\chi_{\lambda/\mu,a}$ be the Jacobi–Trudi-type determinant for the generating series of the (supposed) q-characters of the fundamental representations. We conjecture that $\chi_{\lambda/\mu,a}$ is the q-character of a certain finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$. We study the tableaux description of $\chi_{\lambda/\mu,a}$ using the path method due to Gessel–Viennot. It immediately reproduces the tableau rule by Bazhanov–Reshetikhin for A_n and by Kuniba–Ohta–Suzuki for B_n. For C_n, we derive the explicit tableau rule for skew diagrams λ/μ of three rows and of two columns.

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1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\hat{\mathfrak{g}}$ be the corresponding non-twisted affine Lie algebra. Let $U_q(\hat{\mathfrak{g}})$ be the quantum affine algebra, namely, the quantized universal enveloping algebra of $\hat{\mathfrak{g}}$ [12, 17]. The *q-character* of $U_q(\hat{\mathfrak{g}})$, introduced in [15], is an injective ring homomorphism

$$\chi_q : \mathrm{Rep}(U_q(\hat{\mathfrak{g}})) \to \mathbb{Z}\big[Y_{i,a}^{\pm 1}\big]_{i=1,\ldots,n;a\in\mathbb{C}^\times},$$

where $\mathrm{Rep}(U_q(\hat{\mathfrak{g}}))$ is the Grothendieck ring of the category of the finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$. Like the usual character for $\mathfrak{g}$, $\chi_q(V)$ contains essential data of each representation V. Also, it is a powerful tool to investigate the ring structure of $\mathrm{Rep}(U_q(\hat{\mathfrak{g}}))$. Unfortunately, not much is known about the explicit formula of $\chi_q(V)$ so far.

The q-character is designed to be a 'universalization' of the family of the transfer matrices of the solvable vertex models [5] associated with various R-matrices [6, 18, 19, 27]. The tableaux descriptions of the spectra of the transfer matrices of a vertex model associated with $U_q(\hat{\mathfrak{g}})$ were studied in [7, 20, 22] for $\mathfrak{g}$ of classical type. Then, one can interpret their results in the context of the q-character in the following way: let $\chi_{\lambda/\mu,a}$ be the Jacobi–Trudi determinant (2.23) for the generating series of the (supposed) q-characters of the fundamental representations of $U_q(\hat{\mathfrak{g}})$, where λ/μ is a skew diagram and $a \in \mathbb{C}$. For A_n and B_n, $\chi_{\lambda/\mu,a}$ is conjectured to be the q-character of the finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ associated with λ/μ and a. The determinant $\chi_{\lambda/\mu,a}$ allows the description by the semistandard tableaux of shape λ/μ for A_n [7], and by the tableaux of shape λ/μ which satisfy certain 'horizontal' and 'vertical' rules similar to the rules of the semistandard tableaux for B_n [20] (see definition 4.4 for the rules). For C_n and D_n, we still conjecture (conjecture 2.2) that $\chi_{\lambda/\mu,a}$ is the q-character of a certain, but not necessarily irreducible, representation of $U_q(\hat{\mathfrak{g}})$. However, the tableaux description for $\chi_{\lambda/\mu,a}$ is known only for the basic cases, $(\lambda, \mu) = ((1^i), \phi)$ and $(\lambda, \mu) = ((i), \phi)$ [14, 21, 22].

The main purpose of this paper is to give the tableaux description of $\chi_{\lambda/\mu,a}$ in the C_n case. Let us preview our results and explain what makes the tableaux description more complicated for C_n and D_n than A_n and B_n. To obtain the tableaux description of $\chi_{\lambda/\mu,a}$, we apply the paths method of [16]. The method was originally introduced to derive the well-known semistandard tableaux description of the Schur function from the (original) Jacobi–Trudi determinant; but, the idea is applicable to our determinant $\chi_{\lambda/\mu,a}$, too. Roughly speaking, the method works as follows: first, we express the determinant by sequences of 'paths'. Then, the contributions for the determinant from the intersecting sequences of paths cancel, and we obtain a positive sum expression of the determinant by the nonintersecting sequences of paths. Finally, we translate each nonintersecting sequence of paths into a 'tableau'; the definition of a path and the nonintersecting property turn into the horizontal and vertical rules, respectively. For A_n, the method works perfectly, and it immediately reproduces the result of [7] above. For B_n, though a slight modification is required, it works well too, and reproduces the result of [20] above. For C_n and D_n, however, it turns out that the contributions from the intersecting sequences of the paths do not completely cancel out, and we only get an *alternative* sum expression by nonintersecting and intersecting sequences of paths. Therefore, we need one more step to translate it into a *positive* sum expression by tableaux, and it can be done essentially by the inclusion–exclusion principle. Then, due to the negative contribution in the alternative sum, some additional rules emerge besides the horizontal and vertical rules, which we call the *extra* rules (see the two-row diagram case in section 5.3 for the simplest example). It turns out, however, that these extra rules depend on the shape λ/μ, and have infinitely many varieties. This explains, at least in our point of view, why the tableaux description for C_n and D_n has not been known so far except for the basic cases.

The outline of the paper is as follows. In section 2, we define the Jacobi–Trudi determinant $\chi_{\lambda/\mu,a}$ (2.23) and formulate our basic conjecture (conjecture 2.2) that $\chi_{\lambda/\mu,a}$ is the q-character of an irreducible representation of $U_q(\hat{\mathfrak{g}})$ (for C_n and D_n, $\mu = \phi$). In sections 3 and 4, we show how the Gessel–Viennot method works well to reproduce the results of [7] for A_n and [20] for B_n. In section 5, we consider the C_n case. This is the main part of this paper. As explained above, the Gessel–Viennot method only gives an alternative sum expression $\chi_{\lambda/\mu,a}$ in terms of paths (proposition 5.3). To apply the inclusion–exclusion principle, we introduce the 'resolution' of a transposed pair of paths, and derive the extra rules explicitly for the skew diagram of three rows (theorem 5.7) and of two columns (theorem 5.10 and conjecture 5.9).

For general skew diagrams, the extra rules have infinitely many variety, and so far we have not found a unified way to write them down explicitly. However, the above examples suggest that, after all, the extra rules are better described *in terms of paths*. We plan to study it in a separate publication. The D_n case is similar to C_n, and it will be treated also in a separate publication.

Let us briefly mention two possible applications of the results. Firstly, the affine crystal for the Kirillov–Reshetikhin representations, which are special cases of the representations treated here, are highly expected but known only for basic cases (see [29], for example). It is interesting to examine whether there is a natural affine crystal structure on our tableaux.

Secondly, our tableaux are quite compatible with the conjectural algorithm of [13] to create the q-character. We hope that our tableaux help us to prove the algorithm for these representations and also to prove conjecture 2.2 itself.

2. q-characters and the Jacobi-Trudi determinant

In this section, we give the conjecture of the Jacobi–Trudi-type formula of the q-characters. Throughout this paper, we assume that $q^k \neq 1$ for any $k \in \mathbb{Z}$.

2.1. The variable $Y_{i,a}^{\pm 1}$ and $z_{i,a}$

The q-character is originally described as a polynomial in $\mathbb{Z}\big[Y_{i,a}^{\pm 1}\big]_{i=1,\dots,n;a\in\mathbb{C}^\times}$ in [15], where $Y_{i,a}$ is the affinization of the formal exponential $y_i := \mathrm{e}^{\omega_i}$ of the fundamental weight ω_i in the character $\chi : \mathrm{Rep}\, U_q(\mathfrak{g}) \to \mathbb{Z}\big[y_i^{\pm 1}\big]_{i=1,\dots,n}$ of $U_q(\mathfrak{g})$, with the spectral parameter $a \in \mathbb{C}^\times$. For simplicity, we write the variable $Y_{i,aq^k}^{\pm 1}$ [15] in a 'logarithmic' form as $Y_{i,a'+k}^{\pm 1}$, where $k \in \mathbb{Z}, a' = \log_q a \in \mathbb{C}$ and $q \in \mathbb{C}^\times$. In this subsection, we transform the variables $\big\{Y_{i,a}^{\pm 1}\big\}_{i=1,\dots,n;a\in\mathbb{C}}$ into new variables $\{z_{i,a}\}_{i\in I;a\in\mathbb{C}}$, which represent the monomials in the q-character of the first fundamental representation (see (2.14)).

Set

$$\mathcal{Y} = \begin{cases} \mathbb{Z}\big[Y_{1,a}^{\pm 1}, Y_{2,a}^{\pm 1}, \dots, Y_{n,a}^{\pm 1}\big]_{a\in\mathbb{C}}, & (A_n, C_n) \\ \mathbb{Z}\big[Y_{1,a}^{\pm 1}, Y_{2,a}^{\pm 1}, \dots, Y_{n-1,a}^{\pm 1}, Y_{n,a-1}^{\pm 1}Y_{n,a+1}^{\pm 1}\big]_{a\in\mathbb{C}}, & (B_n) \\ \mathbb{Z}\big[Y_{1,a}^{\pm 1}, Y_{2,a}^{\pm 1}, \dots, Y_{n-2,a}^{\pm 1}, Y_{n-1,a}^{\pm 1}Y_{n,a}^{\pm 1}, Y_{n,a-1}^{\pm 1}Y_{n,a+1}^{\pm 1}\big]_{a\in\mathbb{C}}. & (D_n) \end{cases}$$

Let I be a set of letters,

$$I = \begin{cases} \{1, 2, \dots, n, n+1\}, & (A_n) \\ \{1, 2, \dots, n, 0, \overline{n}, \dots, \overline{2}, \overline{1}\}, & (B_n), \\ \{1, 2, \dots, n, \overline{n}, \dots, \overline{2}, \overline{1}\}, & (C_n, D_n). \end{cases} \tag{2.1}$$

Let $\mathcal{Z}$ be the commutative ring over $\mathbb{Z}$ generated by $\{z_{i,a}\}_{i\in I;a\in\mathbb{C}}$, with the following generating relations ($a \in \mathbb{C}$) with $z_{0,a} = z_{\overline{0},a} = 1$ in (2.4) and (2.5):

$$\prod_{k=1}^{n+1} z_{k,a-2k} = 1, \qquad (A_n) \tag{2.2}$$

$$\begin{cases} z_{i,a}z_{\overline{i},a-4n+4i-2} = z_{i-1,a}z_{\overline{i-1},a-4n+4i-2} & (i = 2, \dots, n), \\ z_{1,a}z_{\overline{1},a-4n+2} = 1, \qquad z_{0,a} = \prod_{k=1}^{n} z_{k,a+4n-4k}z_{\overline{k},a-4n+4k}, \end{cases} \qquad (B_n) \tag{2.3}$$

$$z_{i,a}z_{\overline{i},a-2n+2i-4} = z_{i-1,a}z_{\overline{i-1},a-2n+2i-4} \qquad (i = 1, \dots, n), \qquad (C_n) \tag{2.4}$$

$$z_{i,a}z_{\overline{i},a-2n+2i} = z_{i-1,a}z_{\overline{i-1},a-2n+2i} \qquad (i = 1, \dots, n). \qquad (D_n) \tag{2.5}$$

We have

Proposition 2.1. *$\mathcal{Z}$ is isomorphic to $\mathcal{Y}$ as a ring.*

Proof. Let $f : \mathcal{Z} \to \mathcal{Y}$ be a ring homomorphism defined as follows, with $Y_{0,a} = 1$, and in (2.6), $Y_{n+1,a} = 1$:

$$z_{i,a} \mapsto Y_{i,a+i-1}Y_{i-1,a+i}^{-1}, \qquad i = 1, \dots, n+1, \qquad (A_n) \tag{2.6}$$

$$\begin{cases} z_{i,a} \mapsto Y_{i,a+2i-2}Y^{-1}_{i-1,a+2i}, & i=1,\dots,n-1, \\ z_{n,a} \mapsto Y_{n,a+2n-3}Y_{n,a+2n-1}Y^{-1}_{n-1,a+2n}, \\ z_{\overline{n},a} \mapsto Y_{n-1,a+2n-2}Y^{-1}_{n,a+2n-1}Y^{-1}_{n,a+2n+1}, \\ z_{\overline{i},a} \mapsto Y_{i-1,a+4n-2i-2}Y^{-1}_{i,a+4n-2i}, & i=1,\dots,n-1, \end{cases} \quad (B_n) \tag{2.7}$$

$$\begin{cases} z_{i,a} \mapsto Y_{i,a+i-1}Y^{-1}_{i-1,a+i}, & i=1,\dots,n, \\ z_{\overline{i},a} \mapsto Y_{i-1,a+2n-i+2}Y^{-1}_{i,a+2n-i+3}, & i=1,\dots,n, \end{cases} \quad (C_n) \tag{2.8}$$

$$\begin{cases} z_{i,a} \mapsto Y_{i,a+i-1}Y^{-1}_{i-1,a+i}, & i=1,\dots,n-2, \\ z_{n-1,a} \mapsto Y_{n,a+n-2}Y_{n-1,a+n-2}Y^{-1}_{n-2,a+n-1}, \\ z_{n,a} \mapsto Y_{n,a+n-2}Y^{-1}_{n-1,a+n}, \\ z_{\overline{n},a} \mapsto Y_{n-1,a+n-2}Y^{-1}_{n,a+n}, \\ z_{\overline{n-1},a} \mapsto Y_{n-2,a+n-1}Y^{-1}_{n-1,a+n}Y^{-1}_{n,a+n}, \\ z_{\overline{i},a} \mapsto Y_{i-1,a+2n-i-2}Y^{-1}_{i,a+2n-i-1}, & i=1,\dots,n-2. \end{cases} \quad (D_n) \tag{2.9}$$

Let $g : \mathcal{Y} \to \mathcal{Z}$ be the ring homomorphism defined as follows ($a \in \mathbb{C}$):

$$\begin{cases} Y_{i,a} \mapsto \prod_{k=1}^{i} z_{k,a+i-2k+1}, & i=1,\dots,n, \\ Y^{-1}_{i,a} \mapsto \prod_{k=i+1}^{n+1} z_{k,a+i-2k+1}, & i=1,\dots,n, \end{cases} \quad (A_n) \tag{2.10}$$

$$\begin{cases} Y_{i,a} \mapsto \prod_{k=1}^{i} z_{k,a+2i-4k+2}, & i=1,\dots,n-1, \\ Y_{n,a-1}Y_{n,a+1} \mapsto \prod_{k=1}^{n} z_{k,a+2n-4k+2}, \\ Y^{-1}_{n,a-1}Y^{-1}_{n,a+1} \mapsto \prod_{k=1}^{n} z_{\overline{k},a-6n+4k}, \\ Y^{-1}_{i,a} \mapsto \prod_{k=1}^{i} z_{\overline{k},a-4n-2i+4k}, & i=1,\dots,n-1, \end{cases} \quad (B_n) \tag{2.11}$$

$$\begin{cases} Y_{i,a} \mapsto \prod_{k=1}^{i} z_{k,a+i-2k+1}, & i=1,\dots,n, \\ Y^{-1}_{n,a} \mapsto \prod_{k=1}^{i} z_{\overline{k},a-2n-i+2k-3}, & i=1,\dots,n, \end{cases} \quad (C_n) \tag{2.12}$$

$$\begin{cases} Y_{i,a} \mapsto \prod_{k=1}^{i} z_{k,a+i-2k+1}, & i=1,\dots,n-2, \\ Y_{n-1,a}Y_{n,a} \mapsto \prod_{k=1}^{n-1} z_{k,a+n-2k}, \\ Y_{n,a-1}Y_{n,a+1} \mapsto \prod_{k=1}^{n} z_{k,a+n-2k+1}, \\ Y^{-1}_{n,a-1}Y^{-1}_{n,a+1} \mapsto \prod_{k=1}^{n} z_{\overline{k},a-3n+2k+1}, \\ Y^{-1}_{n-1,a}Y^{-1}_{n,a} \mapsto \prod_{k=1}^{n-1} z_{\overline{k},a-3n+2k+2}, \\ Y^{-1}_{i,a} \mapsto \prod_{k=1}^{i} z_{\overline{k},a-2n-i+2k+1}, & i=1,\dots,n-2. \end{cases} \quad (D_n) \tag{2.13}$$

It is easy to check that each homomorphism is well defined and $f \circ g = g \circ f = \mathrm{id}$, so that f and g are inverse to each other. □

From now, we identify $\mathcal{Y}$ with $\mathcal{Z}$ by the isomorphism f. Then, the q-character of the first fundamental representation $V_{\omega_1}(q^a)$ is given as [15]

$$\chi_q\big(V_{\omega_1}(q^a)\big) = \sum_{i \in I} z_{i,a}. \tag{2.14}$$

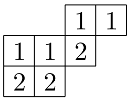


Figure 1. The highest weight tableau of λ/μ for $(\lambda, \mu) = ((4, 3, 2), (2))$.

2.2. Partitions, Young diagrams and tableaux

A *partition* is a sequence of weakly decreasing non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ with finitely many nonzero terms $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_l > 0$. The *length* $l(\lambda)$ of λ is the number of the nonzero integers in λ. The *conjugate* of λ is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$. As usual, we identify a partition λ with a *Young diagram* $\lambda = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leqslant j \leqslant \lambda_i\}$, and also identify a pair of partitions (λ, μ) such that $\mu \subset \lambda$, i.e., $\lambda_i - \mu_i \geqslant 0$ for any i, with a *skew diagram* $\lambda/\mu = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid \mu_i + 1 \leqslant j \leqslant \lambda_i\}$. If $\mu = \phi$, we write a skew diagram as a Young diagram λ instead of λ/ϕ. The *depth* $d(\lambda/\mu)$ of λ/μ is the length of its longest column, i.e., $d(\lambda/\mu) = \max\{\lambda'_i - \mu'_i\}$. A *tableau* T of shape λ/μ is the skew diagram λ/μ with each box filled by one entry of I (2.1).

For a tableau T and $a \in \mathbb{C}$, we define

$$z_a^T := \prod_{(i,j)\in\lambda/\mu} z_{T(i,j),a+2(j-i)\delta}, \tag{2.15}$$

where $T(i, j)$ is the entry of T at (i, j), namely, the entry at the ith row and the jth column, and δ is

$$\delta = \begin{cases} 1, & (A_n, C_n, D_n) \\ 2. & (B_n) \end{cases} \tag{2.16}$$

For any skew diagram λ/μ with $d(\lambda/\mu) \leqslant n$, let T_+ be the tableau of shape λ/μ such that $T(i, j) = i - \mu'_j$ for all $(i, j) \in \lambda/\mu$. We call T_+ the *highest weight tableau* of λ/μ. See figure 1 for example. Then we have

$$f\left(z_a^{T_+}\right) = \prod_{j=1}^{l(\lambda')} Y^{1-\beta(j)}_{\lambda'_j-\mu'_j,a(j)} Y^{\alpha(j)}_{n,a(j)} Y^{\beta(j)}_{n,a(j)-1} Y^{\beta(j)}_{n,a(j)+1}, \tag{2.17}$$

where f is the isomorphism in the proof of proposition 2.1 and

$$a(j) = a + (2j - \lambda'_j - \mu'_j - 1)\delta,$$

$$\alpha(j) = \begin{cases} 1, & \text{if } \mathfrak{g} \text{ is of type } D_n \text{ and } \lambda'_j - \mu'_j = n - 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\beta(j) = \begin{cases} 1, & \text{if } \mathfrak{g} \text{ is of type } B_n \text{ or } D_n \text{ and } \lambda'_j - \mu'_j = n, \\ 0, & \text{otherwise.} \end{cases}$$

2.3. Representations of $U_q(\hat{\mathfrak{g}})$ associated with skew diagrams

There is a bijection between the set of the isomorphism classes of the finite-dimensional irreducible representations of $U_q(\hat{\mathfrak{g}})$ and the set of n-tuples of polynomials [10, 11]

$$\mathbf{P}(u) = (P_i(u))_{i=1,\ldots,n}, \qquad P_i(u) \in \mathbb{C}[u] \quad \text{with constant term 1,}$$

which are called the *Drinfel'd polynomials*. Let $V(\mathbf{P}(u))$ be the representation associated with $\mathbf{P}(u)$, where

$$P_i(u) = \prod_{k=1}^{n_i} (1 - uq^{a_{ik}}), \qquad i = 1, \ldots, n.$$

Then the q-character $\chi_q(V(\mathbf{P}(u)))$ contains the *highest weight monomial*

$$m(\mathbf{P}(u)) := \prod_{i=1}^{n}\prod_{k=1}^{n_i} Y_{i,a_{ik}} \tag{2.18}$$

with multiplicity 1 [13].

For any skew diagram λ/μ with $d(\lambda/\mu) \leqslant n$, one can uniquely associate a finite-dimensional irreducible representation of $U_q(\hat{\mathfrak{g}})$ such that its highest weight monomial (2.18) coincides with (2.17) for the highest weight tableau T_+ of λ/μ. We write this representation as $V(\lambda/\mu, a)$. Namely, $V(\lambda/\mu, a)$ is the representation that corresponds to the Drinfel'd polynomial

$$\prod_{j=1}^{l(\lambda')} \mathbf{P}^{1-\beta(j)}_{\lambda'_j-\mu'_j,a(j)}(u)\mathbf{P}^{\alpha(j)}_{n,a(j)}(u)\mathbf{P}^{\beta(j)}_{n,a(j)-1}(u)\mathbf{P}^{\beta(j)}_{n,a(j)+1}(u),$$

where $\mathbf{PQ} := (P_jQ_j)_{j=1,\dots,n}$ for any $\mathbf{P} = (P_j)_{j=1,\dots,n}$ and $\mathbf{Q} = (Q_j)_{j=1,\dots,n}$, and $\mathbf{P}^{\gamma}_{i,a}(u) = (P_j(u))_{j=1,\dots,n}$ is defined as

$$P_j(u) = \begin{cases} 1-uq^a, & \text{if } j = i \quad \text{and} \quad \gamma = 1, \\ 1, & \text{otherwise.} \end{cases}$$

2.4. The Jacobi–Trudi formula for the q-characters

Let δ be the number in (2.16). Let $\mathbb{Z}[[X]]$ be the formal power series ring over $\mathbb{Z}$ with variable X. Let $\mathcal{A}$ be the *non-commutative* ring generated by $\mathcal{Z}$ and $\mathbb{Z}[[X]]$ with relations

$$Xz_{i,a} = z_{i,a-2\delta}X, \qquad i \in I, \quad a \in \mathbb{C}. \tag{2.19}$$

For any $a \in \mathbb{C}$, we define $E_a(z, X), H_a(z, X) \in \mathcal{A}$ as follows:

$$E_a(z,X) := \begin{cases} \overrightarrow{\prod}_{1\leqslant k\leqslant n+1} (1+z_{k,a}X) & (A_n) \\ \{\overrightarrow{\prod}_{1\leqslant k\leqslant n} (1+z_{k,a}X)\}(1-z_{0,a}X)^{-1}\{\overleftarrow{\prod}_{1\leqslant k\leqslant n} (1+z_{\overline{k},a}X)\} & (B_n) \\ \{\overrightarrow{\prod}_{1\leqslant k\leqslant n} (1+z_{k,a}X)\}(1-z_{n,a}Xz_{\overline{n},a}X)\{\overleftarrow{\prod}_{1\leqslant k\leqslant n} (1+z_{\overline{k},a}X)\} & (C_n) \\ \{\overrightarrow{\prod}_{1\leqslant k\leqslant n} (1+z_{k,a}X)\}(1-z_{\overline{n},a}Xz_{n,a}X)^{-1}\{\overleftarrow{\prod}_{1\leqslant k\leqslant n} (1+z_{\overline{k},a}X)\} & (D_n) \end{cases} \tag{2.20}$$

$$H_a(z,X) := \begin{cases} \overleftarrow{\prod}_{1\leqslant k\leqslant n+1} (1-z_{k,a}X)^{-1} & (A_n) \\ \{\overrightarrow{\prod}_{1\leqslant k\leqslant n} (1-z_{\overline{k},a}X)^{-1}\}(1+z_{0,a}X)\{\overleftarrow{\prod}_{1\leqslant k\leqslant n} (1-z_{k,a}X)^{-1}\} & (B_n) \\ \{\overrightarrow{\prod}_{1\leqslant k\leqslant n} (1-z_{\overline{k},a}X)^{-1}\}(1-z_{n,a}Xz_{\overline{n},a}X)^{-1}\{\overleftarrow{\prod}_{1\leqslant k\leqslant n} (1-z_{k,a}X)^{-1}\} & (C_n) \\ \{\overrightarrow{\prod}_{1\leqslant k\leqslant n} (1-z_{\overline{k},a}X)^{-1}\}(1-z_{\overline{n},a}Xz_{n,a}X)\{\overleftarrow{\prod}_{1\leqslant k\leqslant n} (1-z_{k,a}X)^{-1}\} & (D_n) \end{cases} \tag{2.21}$$

where $\overrightarrow{\prod}_{1\leqslant k\leqslant n} A_k = A_1\cdots A_n$ and $\overleftarrow{\prod}_{1\leqslant k\leqslant n} A_k = A_n\cdots A_1$. Then we have

$$H_a(z,X)E_a(z,-X) = E_a(z,-X)H_a(z,X) = 1. \tag{2.22}$$

For any $i \in \mathbb{Z}$ and $a \in \mathbb{C}$, we define $e_{\mathrm{i},a}, h_{i,a} \in \mathcal{Z}$ as

$$E_a(z,X) = \sum_{i=0}^{\infty} e_{\mathrm{i},a}X^i, \qquad H_a(z,X) = \sum_{i=0}^{\infty} h_{i,a}X^i.$$

Set $e_{i,a} = h_{i,a} = 0$ for $i < 0$. Note that $e_{i,a} = 0$ if $i > n+1$ (resp. if $i > 2n+2$ or $i = n+1$) for A_n (resp. for C_n).

It has been observed in [14, 21] (see also [20, 22]) that $e_{i,a}$ is the q-character of the ith fundamental representation for $1 \leqslant i \leqslant n$ ($i \neq n$ for B_n, $i \neq n-1, n$ for D_n), while $h_{i,a}$ is the q-character of the ith 'symmetric' power of the first fundamental representation for any $i \geqslant 1$, though only a part of them are proven in the literature (e.g. [26]).

Due to relation (2.22), it holds that [25]

$$\det\left(h_{\lambda_i-\mu_j-i+j,a+2(\lambda_i-i)\delta}\right)_{1\leqslant i,j\leqslant l} = \det\left(e_{\lambda'_i-\mu'_j-i+j,a-2(\mu'_j-j+1)\delta}\right)_{1\leqslant i,j\leqslant l'} \quad (2.23)$$

for any partitions (λ, μ), where l and l' are any non-negative integers such that $l \geqslant l(\lambda), l(\mu)$ and $l' \geqslant l(\lambda'), l(\mu')$. For any skew diagram λ/μ, let $\chi_{\lambda/\mu,a}$ denote the determinant on the left- or right-hand side of (2.23). We call it the *Jacobi–Trudi determinant* of $U_q(\hat{\mathfrak{g}})$ associated with λ/μ and $a \in \mathbb{C}$. Note that $\chi_{(i),a} = h_{i,a}$ and $\chi_{(1^i),a} = e_{i,a}$.

Conjecture 2.2.

(1) If $\mathfrak{g}$ is of type A_n or B_n and λ/μ is a skew diagram of $d(\lambda/\mu) \leqslant n$, then $\chi_{\lambda/\mu,a} = \chi_q(V(\lambda/\mu, a))$.

(2) If $\mathfrak{g}$ is of type C_n and λ/μ is a skew diagram of $d(\lambda/\mu) \leqslant n$, then $\chi_{\lambda/\mu,a}$ is the q-character of certain (not necessarily irreducible) representation V of $U_q(\hat{\mathfrak{g}})$ which has $V(\lambda/\mu, a)$ as a subquotient; furthermore, if $\mu = \phi$, then $V = V(\lambda, a)$.

(3) If $\mathfrak{g}$ is of type D_n and λ/μ is a skew diagram of $d(\lambda/\mu) \leqslant n$, then $\chi_{\lambda/\mu,a}$ is the q-character of certain (not necessarily irreducible) representation V of $U_q(\hat{\mathfrak{g}})$ which has $V(\lambda/\mu, a)$ as a subquotient; furthermore, if $\mu = \phi$ and $d(\lambda) \leqslant n-1$, then $V = V(\lambda, a)$.

Several remarks on conjecture 2.2 are in order.

(1) For C_n, we checked by computer that $\chi_{\lambda,a}$ agrees with the result obtained from the conjectural algorithm of [13] to create the q-character for several λ.

(2) It is interesting that the determinant (2.23) is simpler than the Jacobi–Trudi-type formula for the characters of $\mathfrak{g}$ for the irreducible representations $V(\lambda)$ in [23].

(3) The determinant $\chi_{\lambda/\mu,a}$ appeared in [7] for A_n and [20] for B_n in the context of the transfer matrices.

(4) An analogue of conjecture 2.2 is true for the representations of Yangian $Y(\mathfrak{sl}_n)$, which can be proved [2] using the results in [3, 4].

(5) Conjecture 2.2 is an affinization of the conjecture of [9] (see remark A.2 in appendix A).

(6) For C_n and D_n, we further expect that $V = V(\lambda/\mu, a)$ if λ/μ is connected. But, if λ/μ is not connected, there are certainly counter-examples. A counter-example for C_2 is as follows: let $(\lambda, \mu) = ((3,1),(2))$. By (2.23), we have $\chi_{\lambda/\mu,a+2} = h_{1,a}h_{1,a+6} = \chi_q(V_{\omega_1}(q^a) \otimes V_{\omega_1}(q^{a+6}))$. On the other hand, the R-matrix $R_{\omega_1,\omega_1}(u)$ has singularities at $u = q^6$ (see [1] for example), which implies that $V_{\omega_1}(q^a) \otimes V_{\omega_1}(q^{a+6})$ is not irreducible. The case $(\lambda, \mu) = ((3,1),(2))$ for D_4 is a similar counter-example.

In the following sections, we study the explicit description of $\chi_{\lambda/\mu,a}$ by tableaux.

3. Tableaux description of type A_n

In this section, we consider the case that $\mathfrak{g}$ is of type A_n. The tableaux description of $\chi_{\lambda/\mu,a}$ (2.23) is given by [7]. We reproduce it by applying the 'paths' method of [16] (see also [28]). During this section, I is of type A_n in (2.1).

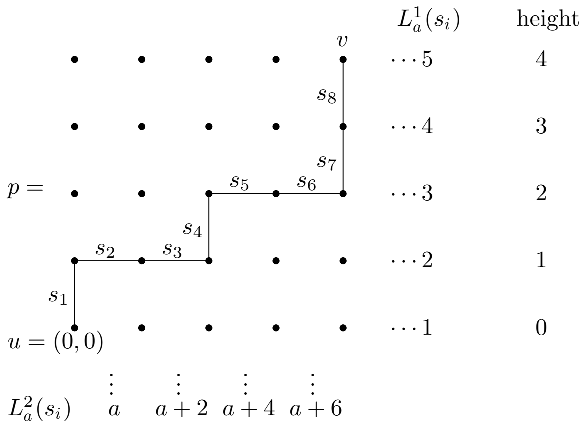


Figure 2. An example of a path p and its h-labelling.

3.1. Paths description

Consider the lattice $\mathbb{Z} \times \mathbb{Z}$. A *path* p in the lattice is a sequence of steps $(s_1, s_2, \ldots)$ such that each step s_i is of unit length with the northward (N) or eastward (E) direction. For example, see figure 2. If p starts at point u and ends at point v, we write this by $u \stackrel{p}{\to} v$. For any path p, set $E(p) := \{s \in p \mid s \text{ is an eastward step}\}$.

An *h-path of type* A_n is a path $u \stackrel{p}{\to} v$ such that the initial point u is at height 0 and the final point v is at height n, where the *height* of the point $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is y. Let $P(A_n)$ be the set of all the h-paths of type A_n. For any $a \in \mathbb{C}$, the *h-labelling of type* A_n associated with $a \in \mathbb{C}$ for a path $p \in P(A_n)$ is the pair of maps $L_a = \left(L^1_a, L^2_a\right)$,

$$L^1_a : E(p) \to I, \qquad L^2_a : E(p) \to \{a + 2k \mid k \in \mathbb{Z}\},$$

defined as follows: if s starts at the point (x, y), then $L_a(s) = (y + 1, a + 2x)$. For example, $L_a(s_3) = (2, a + 2)$ for s_3 in figure 2. Using these definitions, we define

$$z^p_a = \prod_{s \in E(p)} z_{L^1_a(s), L^2_a(s)} \in \mathcal{Z} \tag{3.1}$$

for any $p \in P(A_n)$, where $\mathcal{Z}$ is the ring defined in section 2. For example, $z^p_a = z_{2,a} z_{2,a+2} z_{3,a+4} z_{3,a+6}$ for p in figure 2. By (2.21), we have

$$h_{r,a+2k+2r-2}(z) = \sum_p z^p_a, \tag{3.2}$$

where the sum runs over all $p \in P(A_n)$ such that $(k, 0) \stackrel{p}{\to} (k + r, n)$.

For any l-tuples of initial points $\mathbf{u} = (u_1, u_2, \ldots, u_l)$ and final points $\mathbf{v} = (v_1, v_2, \ldots, v_l)$, let $\mathfrak{P}(\pi; \mathbf{u}, \mathbf{v})$ be the set of l-tuples of paths $\mathbf{p} = (p_1, \ldots, p_l)$ such that $u_i \stackrel{p_i}{\to} v_{\pi(i)}$ for any permutation $\pi \in \mathfrak{S}_l$. Set

$$\mathfrak{P}(\mathbf{u}, \mathbf{v}) := \sum_{\pi \in \mathfrak{S}_l} \mathfrak{P}(\pi; \mathbf{u}, \mathbf{v}).$$

Then we define

$$\mathfrak{P}(A_n; \mathbf{u}, \mathbf{v}) := \{\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(\mathbf{u}, \mathbf{v}) \mid p_i \in P(A_n)\}.$$

For any skew diagram λ/μ, let $l = l(\lambda)$, and pick $\mathbf{u}_\mu = (u_1, \ldots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \ldots, v_l)$ as $u_i = (\mu_i + 1 - i, 0)$ and $v_i = (\lambda_i + 1 - i, n)$. In this case, we have $\mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda) = \mathfrak{P}(\mathbf{u}_\mu, \mathbf{v}_\lambda)$. We define the *weight* $z_a^{\mathbf{p}}$ and the *signature* $(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ by

$$z_a^{\mathbf{p}} = \prod_{i=1}^{l} z_a^{p_i} \qquad \text{and} \qquad (-1)^{\mathbf{p}} = \operatorname{sgn} \pi \qquad \text{if} \quad \mathbf{p} \in \mathfrak{P}(\pi; \mathbf{u}_\mu, \mathbf{v}_\lambda). \tag{3.3}$$

Then, determinant (2.23) can be written as

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}}, \tag{3.4}$$

by (3.2). Applying the method of [16], we have

Proposition 3.1. *For any skew diagram λ/μ,*

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in P(A_n; \mu, \lambda)} z_a^{\mathbf{p}}, \tag{3.5}$$

where $P(A_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any intersecting pair of paths (p_i, p_j).

Proof. Let $P^c(A_n; \mu, \lambda) := \{\mathbf{p} \in \mathfrak{P}(A_n; \mathbf{u}_\mu, \mathbf{v}_\lambda) \mid \mathbf{p} \notin P(A_n; \mu, \lambda)\}$. The idea of [16] is to consider an involution

$$\iota : P^c(A_n; \mu, \lambda) \to P^c(A_n; \mu, \lambda)$$

defined as follows: for $\mathbf{p} = (p_1, \ldots, p_l)$, let (p_i, p_j) be the first intersecting pair of paths, i.e., i is the minimal number such that p_i intersects with another path and j $(\neq i)$ is the minimal number such that p_j intersects with p_i. Let v_0 be the first intersecting point of p_i and p_j. If $u_i \xrightarrow{p_i} v_{\pi(i)}$ $(i = 1, \ldots, l)$, then $\iota(\mathbf{p}) = (p'_1, \ldots, p'_l)$ is given by $p'_k := p_k$ $(k \neq i, j)$ and

$$p'_i : u_i \xrightarrow{p_i} v_0 \xrightarrow{p_j} v_{\pi(j)}, \qquad p'_j : u_j \xrightarrow{p_j} v_0 \xrightarrow{p_i} v_{\pi(i)}.$$

Then ι preserves the weights and inverts the signature, i.e., $z_a^{\iota(\mathbf{p})} = z_a^{\mathbf{p}}$ and $(-1)^{\iota(\mathbf{p})} = -(-1)^{\mathbf{p}}$. Therefore, the contributions of all $\mathbf{p} \in P^c(A_n; \mu, \lambda)$ to the right-hand side of (3.4) are cancelled with each other. The signature of any $\mathbf{p} \in P(A_n; \mu, \lambda)$ is $(-1)^{\mathbf{p}} = 1$, and we obtain the proposition. □

3.2. Tableaux description

Definition 3.2. *A tableau T with entries $T(i, j) \in I$ is called an A_n-tableau if it satisfies the following conditions:*

(H)	horizontal rule	$T(i, j) \leqslant T(i, j+1)$.
(V)	vertical rule	$T(i, j) < T(i+1, j)$.

Namely, an A_n-tableau is nothing but a semistandard tableau. We write the set of all the A_n-tableaux of shape λ/μ by $\mathrm{Tab}(A_n, \lambda/\mu)$.

For any $\mathbf{p} = (p_1, \ldots, p_l) \in P(A_n; \mu, \lambda)$, we associate a tableau $T(\mathbf{p})$ of shape λ/μ such that the ith row of $T(\mathbf{p})$ is given by $\{L_a^1(s) \mid s \in E(p_i)\}$ listed in the increasing order. See figure 3 for an example. Clearly, $T(\mathbf{p})$ satisfies the horizontal rule because of the h-labelling rule of $\mathbf{p}$, and $T(\mathbf{p})$ satisfies the vertical rule since $\mathbf{p} \in P(A_n; \mu, \lambda)$ does not have any intersecting pair of paths. Therefore, we obtain a map

$$T : P(A_n; \mu, \lambda) \ni \mathbf{p} \mapsto T(\mathbf{p}) \in \mathrm{Tab}(A_n, \lambda/\mu)$$

for any skew diagram λ/μ. In fact,

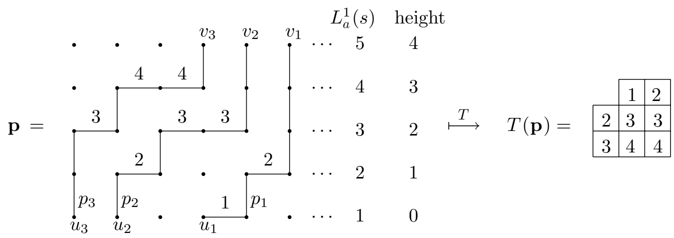


Figure 3. An example of $\mathbf{p}$ and the tableau $T(\mathbf{p})$ for $(\lambda, \mu) = ((3^3), (1))$.

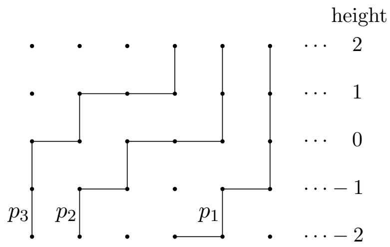


Figure 4. An example of h-paths of types B_n and C_n.

Proposition 3.3. *The map T is a weight-preserving bijection.*

By propositions 3.1 and 3.3, we reproduce the result of [7].

Theorem 3.4 ([7])**.** *If λ/μ is a skew diagram, then*

$$\chi_{\lambda/\mu,a} = \sum_{T \in \mathrm{Tab}(A_n,\lambda/\mu)} z_a^T.$$

4. Tableaux description of type B_n

In this section, we consider the case that $\mathfrak{g}$ is of type B_n. The tableaux description of $\chi_{\lambda/\mu,a}$ (2.23) is given by [20]. We reproduce it using the path method of [16]. During this section, I is of type B_n in (2.1).

4.1. Paths description

In view of the definition of the generating function of $H_a(z, X)$ in (2.21), we define an h-path and its h-labelling as follows:

Definition 4.1. *Consider the lattice $\mathbb{Z} \times \mathbb{Z}$. An h-path of type B_n is a path $u \xrightarrow{p} v$ such that the initial point u is at height $-n$ and the final point v is at height n, and an eastward step at height 0 occurs at most once. We write the set of all the h-paths of type B_n by $P(B_n)$.*

For example, p_1 and p_3 in figure 4 are the h-paths of type B_n, but p_2 is not. The *h-labelling of type B_n* associated with $a \in \mathbb{C}$ for any $p \in P(B_n)$ is the pair of maps $L_a = \left(L_a^1, L_a^2\right)$,

$$L_a^1 : E(p) \to I, \qquad L_a^2 : E(p) \to \{a + 4k \mid k \in \mathbb{Z}\},$$

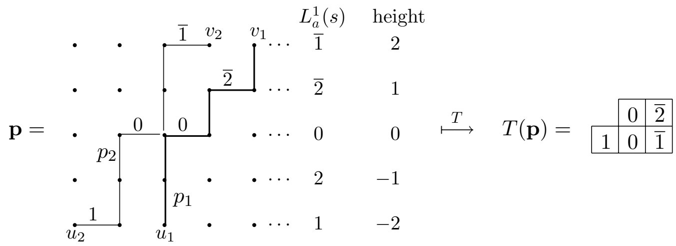


Figure 5. An example of $\mathbf{p} = (p_1, p_2) \in P(B_n; \lambda, \mu)$ which is *specially* intersecting, and their h-labellings for $n = 2$, $(\lambda, \mu) = ((3^2), (1))$. If we set ι as in the A_n case, then $\iota(\mathbf{p}) \notin P(B_n; \mu, \lambda)$, because the paths in $\iota(\mathbf{p})$ are not the h-paths of type B_n.

defined as follows: if s starts at (x, y) then

$$L_a^1(s) = \begin{cases} n+1+y, & \text{if} \quad y < 0, \\ 0, & \text{if} \quad y = 0, \\ \overline{n+1-y}, & \text{if} \quad y > 0, \end{cases}$$

and $L_a^2(s) = a + 4x$. Then, we define z_a^p as in (3.1). By (2.21), we have

$$h_{r,a+4k+4r-4}(z) = \sum_p z_a^p, \tag{4.1}$$

where the sum runs over all $p \in P(B_n)$ such that $(k, -n) \xrightarrow{p} (k + r, n)$.

For any l-tuples of initial and final points $\mathbf{u} = (u_1, u_2, \ldots, u_l)$, $\mathbf{v} = (v_1, v_2, \ldots, v_l)$, set

$$\mathfrak{P}(B_n; \mathbf{u}, \mathbf{v}) := \{\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(\mathbf{u}, \mathbf{v}) \mid p_i \in P(B_n)\}.$$

Let λ/μ be a skew diagram and let $l = l(\lambda)$. Pick $\mathbf{u}_\mu = (u_1, \ldots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \ldots, v_l)$ as $u_i = (\mu_i + 1 - i, -n)$ and $v_i = (\lambda_i + 1 - i, n)$. We define the weight $z^{\mathbf{p}}$ and its signature $(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(B_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$, as in the A_n case in (3.3). Then, determinant (2.23) can be written as

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in \mathfrak{P}(B_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}},$$

by (4.1). The difference from the A_n case is that the involution ι is not defined on any $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(B_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ that possesses an intersecting pair (p_i, p_j) (see figure 5). To define an involution for the B_n case, we give the following definition.

Definition 4.2. *An intersecting pair (p, p') of h-paths of type B_n is called specially intersecting (resp. ordinarily intersecting) if the intersection of p and p' occurs only at height* 0 *(resp. otherwise).*

For example, the pair (p_1, p_2) given in figure 5 is specially intersecting. Applying the method of [16] as in the A_n case, we have

Proposition 4.3. *For any skew diagram λ/μ,*

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p} \in P(B_n; \mu, \lambda)} z_a^{\mathbf{p}}, \tag{4.2}$$

where $P(B_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(B_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any ordinarily intersecting pairs of paths (p_i, p_j).

Proof. Let $P^c(B_n;\mu,\lambda) := \{\mathbf{p} \in \mathfrak{P}(B_n;\mathbf{u}_\mu,\mathbf{v}_\lambda) \mid \mathbf{p} \notin P(B_n;\mu,\lambda)\}$. Consider an involution

$$\iota : P^c(B_n;\mu,\lambda) \to P^c(B_n;\mu,\lambda)$$

defined as follows: for $\mathbf{p} = (p_1,\ldots p_l)$, there exists (p_i, p_j) which is ordinarily intersecting. Let (p_i, p_j) be the first such pair and let v_0 be the first intersecting point whose height is not 0. Then set $\iota(\mathbf{p})$ as in the proof of proposition 3.1. Then ι is weight-preserving and sign-inverting, which implies that all $\mathbf{p} \in P^c(B_n;\mu,\lambda)$ will be cancelled as in the A_n case. The signature of any $\mathbf{p} \in P(B_n;\mu,\lambda)$ is $(-1)^{\mathbf{p}} = 1$, and we obtain the proposition. □

4.2. Tableaux description

Define a total ordering in I (2.1) by

$$1 \prec 2 \prec \ldots \prec n \prec 0 \prec \overline{n} \prec \ldots \prec \overline{2} \prec \overline{1}.$$

Definition 4.4 ([20]). *A tableau T with entries $T(i,j) \in I$ is called a B_n-tableau if it satisfies the following conditions:*

$$\textbf{(H)}\quad T(i,j) \preceq T(i,j+1) \quad \text{and} \quad (T(i,j), T(i,j+1)) \neq (0,0).$$
$$\textbf{(V)}\quad T(i,j) \prec T(i+1,j) \quad \text{or} \quad (T(i,j), T(i+1,j)) = (0,0).$$

We write the set of all the B_n-tableaux of shape λ/μ by $\mathrm{Tab}(B_n,\lambda/\mu)$.

For any $\mathbf{p} \in P(B_n;\mu,\lambda)$, let $T(\mathbf{p})$ be the tableau of shape λ/μ defined by assigning the h-labelling of each path in $\mathbf{p}$ to the corresponding rows, as in the A_n case (see figure 5). Then $T(\mathbf{p})$ satisfies the rule (**H**) in definition 4.4 because of the rule for the h-labelling of $\mathbf{p}$, and it satisfies the rule (**V**) since $\mathbf{p}$ does not have any ordinarily intersecting pairs of paths. Therefore, we obtain a map

$$T : P(B_n;\mu,\lambda) \ni \mathbf{p} \longmapsto T(\mathbf{p}) \in \mathrm{Tab}(B_n,\lambda/\mu)$$

for any skew diagram λ/μ. In fact,

Proposition 4.5. *The map T is a weight-preserving bijection.*

Thus, we obtain

Theorem 4.6 ([20]). *If λ/μ is a skew diagram , then*

$$\chi_{\lambda/\mu,a} = \sum_{T \in \mathrm{Tab}(B_n,\lambda/\mu)} z_a^T.$$

5. Tableaux description of type C_n

In this section, we consider the case that $\mathfrak{g}$ is of type C_n. We determine the tableaux description by the horizontal, vertical and 'extra' rules for skew diagrams of at most three rows and of at most two columns. The one-row and one-column cases are already given by [22].

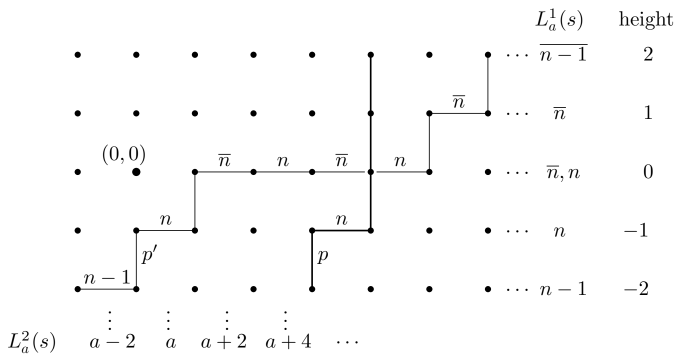


Figure 6. An example of h-paths of type C_n and their h-labellings.

5.1. *Paths description*

In view of the definition of the generating function of $H_a(z, X)$ in (2.21), we define an h-path and its h-labelling as follows:

Definition 5.1. *Consider the lattice $\mathbb{Z} \times \mathbb{Z}$. An h-path of type C_n is a path $u \xrightarrow{p} v$ such that the initial point u is at height $-n$ and the final point v is at height n, and the number of the eastward steps at height 0 is even. We write the set of all the h-paths of type C_n by $P(C_n)$.*

For example, p_1 and p_2 in figure 4 are the h-paths of type C_n, but p_3 is not. For a path $p = (s_1, s_2, \ldots) \in P(C_n)$, let $E_0(p) = (s_j, s_{j+1}, \ldots)$ be the sequence of all the eastward steps at height 0 in p. Let $E_0^1(p)$ and $E_0^2(p)$ be the subsequence of $E_0(p)$ defined by $E_0^1(p) = (s_j, s_{j+2}, s_{j+4}, \ldots)$ and $E_0^2(p) = (s_{j+1}, s_{j+3}, s_{j+5}, \ldots)$. The *h-labelling of type C_n* associated with $a \in \mathbb{C}$ for any $p \in P(C_n)$ is a pair of maps $L_a = \left(L_a^1, L_a^2\right)$,

$$L_a^1 : E(p) \to I, \qquad L_a^2 : E(p) \to \{a + 2k \mid k \in \mathbb{Z}\}, \tag{5.1}$$

defined as follows: if s starts at (x, y), then

$$L_a^1(s) = \begin{cases} n + 1 + y, & \text{if} \quad y < 0, \\ \overline{n + 1 - y}, & \text{if} \quad y > 0, \\ \overline{n}, & \text{if} \quad s \in E_0^1(p), \\ n, & \text{if} \quad s \in E_0^2(p), \end{cases}$$

and $L_a^2(s) = a + 2x$. See figure 6 for an example.

Define z_a^p as in (3.1). By (2.21), we have

$$h_{r,a+2k+2r-2}(z) = \sum_p z_a^p, \tag{5.2}$$

where the sum runs over all $p \in P(C_n)$ such that $(k, -n) \xrightarrow{p} (k + r, n)$.

For any l-tuples of initial and final points $\mathbf{u} = (u_1, u_2, \ldots, u_l)$, $\mathbf{v} = (v_1, v_2, \ldots, v_l)$, set

$$\mathfrak{P}(C_n; \mathbf{u}, \mathbf{v}) := \{\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(\mathbf{u}, \mathbf{v}) \mid p_i \in P(C_n)\}.$$

Let λ/μ be a skew diagram and let $l = l(\lambda)$. Pick $\mathbf{u}_\mu = (u_1, \ldots, u_l)$ and $\mathbf{v}_\lambda = (v_1, \ldots, v_l)$ as $u_i = (\mu_i + 1 - i, -n)$ and $v_i = (\lambda_i + 1 - i, n)$. We define the weight $z^{\mathbf{p}}$ and the signature

$(-1)^{\mathbf{p}}$ for any $\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}, \mathbf{v})$ by the h-labelling of type C_n as in (3.3). Then, determinant (2.23) can be written as

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p}\in\mathfrak{P}(C_n;\mathbf{u}_\mu,\mathbf{v}_\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}},$$

by (5.2).

As in the B_n case, the involution ι is not defined on any $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which possesses an intersecting pair of paths (p_i, p_j). To define the involution for the C_n case, we give the definition of the specially (resp. ordinarily) intersecting pair of paths.

Consider two paths p, p' which are intersecting at height 0. Let $(x, 0)$ (resp. $(x', 0)$) be the leftmost point on p (resp. p') at height 0. Then set $[p, p'] := |x - x'|$.

Definition 5.2. *An intersecting pair (p, p') of h-paths of type C_n is called specially intersecting (resp. ordinarily intersecting) if the intersection of p and p' occurs only at height 0 and $[p, p']$ is odd (resp. otherwise).*

For example, $[p, p'] = 3$ for (p, p') in figure 6, and therefore, it is specially intersecting. Applying the method of [16], we have

Proposition 5.3. *For any skew diagram λ/μ,*

$$\chi_{\lambda/\mu,a} = \sum_{\mathbf{p}\in P(C_n;\mu,\lambda)} (-1)^{\mathbf{p}} z_a^{\mathbf{p}}, \tag{5.3}$$

where $P(C_n; \mu, \lambda)$ is the set of all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any ordinarily intersecting pair of paths (p_i, p_j).

Proof. Let $P^c(C_n; \mu, \lambda) := \{\mathbf{p} \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda) \mid \mathbf{p} \notin P(C_n; \mu, \lambda)\}$. Consider a weight-preserving involution

$$\iota : P^c(C_n; \mu, \lambda) \to P^c(C_n; \mu, \lambda)$$

defined as follows: for $\mathbf{p} = (p_1, \ldots, p_l)$, let (p_i, p_j) be the first ordinarily intersecting pair of paths and let v_0 be the first intersecting point. Set $\iota(\mathbf{p})$ as in the A_n case (proposition 3.1). Then ι is weight-preserving and sign-inverting, as in the A_n and B_n cases. □

Consider any two h-paths of type C_n, $(x_0, y_0) \xrightarrow{p} (x_1, y_1)$, $(x'_0, y'_0) \xrightarrow{p'} (x'_1, y'_1)$, which are not ordinarily intersecting. We say that (p, p') is *transposed* if $(x_0 - x'_0)(x_1 - x'_1) < 0$. For example, the pair (p, p') in figure 6 is transposed.

Let $P_k(C_n; \mu, \lambda)$ be the set of all $\mathbf{p} \in P(C_n; \mu, \lambda)$ which possess exactly k transposed pairs of paths. Then we have $(-1)^{\mathbf{p}} = (-1)^k$. Note that if $\mathbf{p} = (p_1, \ldots p_l) \in P(C_n; \mu, \lambda)$, then each triplet (p_i, p_j, p_k) is not intersecting simultaneously at one point. Therefore, $P(C_n; \mu, \lambda) = \sum_{k=0}^{l-1} P_k(C_n; \mu, \lambda)$ and the sum (5.3) is rewritten as

$$\chi_{\lambda/\mu,a} = \sum_{k=0}^{l-1} (-1)^k \sum_{\mathbf{p}\in P_k(C_n;\mu,\lambda)} z_a^{\mathbf{p}}. \tag{5.4}$$

The right-hand side of (5.4) is not as simple as that of (3.5) for A_n and that of (4.2) for B_n. This is the main reason why the description of C_n becomes more complicated than that of A_n and B_n. (The D_n case which is not dealt with in this paper is similar to the C_n case.)

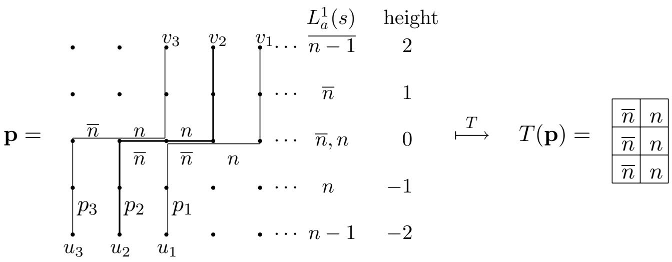


Figure 7. An example of $\mathbf{p} \in \tilde{P}(C_n; \mu, \lambda)$ for $(\lambda, \mu) = ((2^3), \phi)$ and its h-labelling. The pair (p_1, p_3) is ordinarily intersecting, and therefore, $\mathbf{p} \notin P(C_n; \mu, \lambda)$.

5.2. Tableaux description

To formulate the tableaux description of (5.4) we introduce a certain set of tableaux (called HV-tableaux) $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$ and the corresponding set of paths $\tilde{P}(C_n; \mu, \lambda)$.

Define a total ordering in I (2.1) by

$$1 \prec 2 \prec \cdots \prec n \prec \overline{n} \prec \cdots \prec \overline{2} \prec \overline{1}.$$

Definition 5.4. *A tableau T (of shape λ/μ) with entries $T(i, j) \in I$ is called an HV-tableau if it satisfies the following conditions:*
(**H**) *Each $(i, j) \in \lambda/\mu$ satisfies both of the following conditions:*

- $T(i, j) \preceq T(i, j+1)$ *or* $(T(i, j), T(i, j+1)) = (\overline{n}, n)$.
- $(T(i, j-1), T(i, j), T(i, j+1)) \neq (\overline{n}, \overline{n}, n), (\overline{n}, n, n)$.

(**V**) *Each $(i, j) \in \lambda/\mu$ satisfies at least one of the following conditions:*

- $T(i, j) \prec T(i+1, j)$.
- $T(i, j) = T(i+1, j) = n$, $(i+1, j-1) \in \lambda/\mu$ *and* $T(i+1, j-1) = \overline{n}$.
- $T(i, j) = T(i+1, j) = \overline{n}$, $(i, j+1) \in \lambda/\mu$ *and* $T(i, j+1) = n$.

We write the set of the all the HV-tableaux of shape λ/μ by $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$.

Let $\tilde{P}(C_n; \mu, \lambda)$ be the set of all $\mathbf{p} = (p_1, \ldots, p_l) \in \mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ which do not have any *adjacent* pair (p_i, p_{i+1}) which is either ordinarily intersecting or transposed. We remark that $P_k(C_n; \mu, \lambda) \cap \tilde{P}(C_n; \mu, \lambda) = \phi$ $(k \geqslant 1)$ and

$$P_0(C_n; \mu, \lambda) = \tilde{P}(C_n; \mu, \lambda) \qquad \text{if} \quad l(\lambda) \leqslant 2,$$

$$P_0(C_n; \mu, \lambda) \subsetneqq \tilde{P}(C_n; \mu, \lambda) \qquad \text{if} \quad l(\lambda) \geqslant 3.$$

For any $\mathbf{p} \in \tilde{P}(C_n; \mu, \lambda)$, let $T(\mathbf{p})$ be the tableau of shape λ/μ defined by assigning the h-labelling of each path in $\mathbf{p}$ to the corresponding row, as in the A_n and B_n cases (see figure 7). Then $T(\mathbf{p})$ is an HV-tableau; it satisfies the rule (**H**) in definition 5.4 because of the rule for the h-labelling of $\mathbf{p}$, and it satisfies the rule (**V**) since $\mathbf{p}$ does not have any adjacent pairs of paths which are either ordinarily intersecting or transposed. Therefore, we obtain a map

$$T : \tilde{P}(C_n; \mu, \lambda) \ni \mathbf{p} \mapsto T(\mathbf{p}) \in \widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$$

for any skew diagram λ/μ as similar to the previous cases. Moreover,

Proposition 5.5. *The map T is a weight-preserving bijection.*

We expect that the alternative sum (5.4) can be translated into the following positive sum by tableaux,

$$\chi_{\lambda/\mu,a} = \sum_{T \in \mathrm{Tab}(C_n,\lambda/\mu)} z_a^T, \tag{5.5}$$

where $\mathrm{Tab}(C_n, \lambda/\mu)$ is a certain subset of $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$. Thus, the tableaux in $\mathrm{Tab}(C_n, \lambda/\mu)$ are described by the horizontal rules (**H**) and the vertical rules (**V**) in definition 5.4, *and* the *extra* rules which select them out of $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$.

In the following subsections, we show how the tableaux description (5.5) is naturally obtained from (5.4) for the skew diagrams λ/μ of at most three rows and of at most two columns.

Roughly speaking, the idea is as follows (see (5.10) and (5.20)): we introduce the weight-preserving maps f_k which 'resolve' the intersection of a transposed pair of $\mathbf{p} \in P_k(C_n; \mu, \lambda)$ in (5.4), and show that the contributions for (5.4) from $P_k(C_n; \mu, \lambda)$ $(k \geqslant 1)$ almost cancel with each other. Then, the remaining positive contributions fill the difference $\tilde{P}(C_n; \mu, \lambda) \setminus P_0(C_n; \mu, \lambda)$, while the remaining negative contributions turn into the extra rules. We remark that the relation (2.4) plays a crucial role in the weight-preserving property of the maps f_k.

5.3. Skew diagrams of at most three rows

In this subsection, we consider the tableaux description for skew diagrams of at most three rows.

The case of one-row. Let λ/μ be a one-row diagram, i.e., $l(\lambda) = 1$. Then there does not exist any $\mathbf{p} \in P(C_n; \mu, \lambda)$ which possesses a transposed pair of paths, and therefore, $P(C_n; \mu, \lambda) = P_0(C_n; \mu, \lambda) = \tilde{P}(C_n; \mu, \lambda)$. Thus, $\mathrm{Tab}(C_n, \lambda/\mu)$ in equality (5.5) is exactly the set $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$.

The case of two-row. Let λ/μ be a skew diagram of two rows, i.e., $l(\lambda) = 2$. Let $\mathrm{Tab}(C_n, \lambda/\mu)$ be the set of all the HV-tableaux T with the following extra condition:

(**E-2R**) If T contains a subtableau (excluding a and b)

$$\begin{array}{c} \overbrace{\hphantom{n\;n\;\cdots\;n}}^{k} \\ \begin{array}{c|c|c|c|c|c} \multicolumn{1}{c}{} & \hline n & n & \cdots & n & \multicolumn{1}{|c}{a} \\ \hline b & \overline{n} & \overline{n} & \cdots & \overline{n} & \multicolumn{1}{|c}{} \\ \cline{2-5} \end{array} \end{array} \tag{5.6}$$

where k is an odd number, then at least one of the following conditions holds:

(1) Let (i_1, j_1) be the position of the top-right corner of subtableau (5.6). Then $(i_1, j_1 + 1) \in \lambda/\mu$ and $a := T(i_1, j_1 + 1) = n$.
(2) Let (i_2, j_2) be the position of the bottom-left corner of subtableau (5.6). Then $(i_2, j_2 - 1) \in \lambda/\mu$ and $b := T(i_2, j_2 - 1) = \overline{n}$.

Then

Theorem 5.6. *For any skew diagram λ/μ with $l(\lambda) = 2$, equality (5.5) holds.*

Proof. For λ/μ is a skew diagram of $l(\lambda) = 2$, there does not exist any $\mathbf{p} \in P(C_n; \mu, \lambda)$ that possesses more than one transposed pair of paths, we have $P(C_n; \mu, \lambda) = P_0(C_n; \mu, \lambda) \sqcup P_1(C_n; \mu, \lambda)$. We also have $\tilde{P}(C_n; \mu, \lambda) = P_0(C_n; \mu, \lambda)$. Define $f_1 : P_1(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda)$ by $f_1 = r_0$, where r_0 is a weight-preserving injection defined as in appendix B.

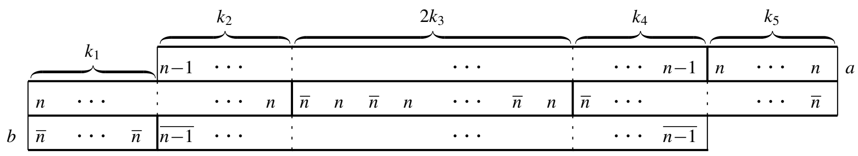
See also figure 11. Roughly speaking, r_0 is a map which resolves the intersection of specially intersecting paths. From (5.4) and proposition 5.5, we have

$$\chi_{\lambda/\mu,a} = \sum_{T \in \widetilde{\mathrm{Tab}}(C_n,\lambda/\mu)} z_a^T - \sum_{\mathbf{p}\in \mathrm{Im}\, f_1} z_a^{T(\mathbf{p})}.$$

The set $\{T(\mathbf{p}) \mid \mathbf{p} \in \mathrm{Im}\, f_1\}$ consists of all the tableaux in $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$ prohibited by the extra rule (**E-2R**). □

The case of three-row. Let λ/μ be a skew diagram of three rows, i.e., $l(\lambda) = 3$. Let $\mathrm{Tab}(C_n, \lambda/\mu)$ be all the HV-tableaux T which satisfy (**E-2R**) and the following conditions: (**E-3R**)

(1) If T contains a subtableau (excluding a and b)



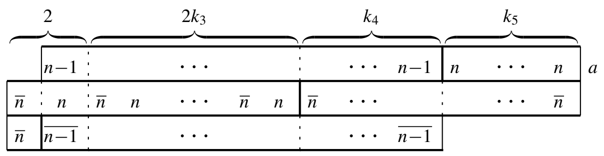
(5.7)

where $k_i \in \mathbb{Z}_{\geqslant 0}$ and $k_1 + k_2 + k_4 + k_5$ is an odd number with $k_2 \neq 0$ or $k_4 \neq 0$, then at least one of the following conditions holds.

(a) Let (i_1, j_1) be the position of the top-right corner of subtableau (5.7). Then $(i_1, j_1 + 1) \in \lambda/\mu$ and $a := T(i_1, j_1 + 1) \prec T(i_1 + 1, j_1)$.

(b) Let (i_2, j_2) be the position of the bottom-left corner of subtableau (5.7). Then $(i_2, j_2 - 1) \in \lambda/\mu$ and $b := T(i_2, j_2 - 1) \succ T(i_2 - 1, j_2)$.

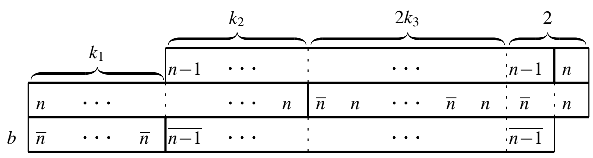
(2) If T contains the subtableau (excluding a)



(5.8)

where $k_i \in \mathbb{Z}_{\geqslant 0}$ and $k_4 + k_5$ is an odd number with $k_4 \neq 0$, then the following holds: Let (i, j) be the position of the top-right corner of subtableau (5.8). Then $(i, j + 1) \in \lambda/\mu$ and $a := T(i, j + 1) \prec T(i + 1, j)$.

(3) If T contains the subtableau (excluding b)



(5.9)

where $k_i \in \mathbb{Z}_{\geqslant 0}$ and $k_1 + k_2$ is an odd number with $k_2 \neq 0$, then the following holds: Let (i, j) be the position of the bottom-left corner of subtableau (5.9). Then $(i, j - 1) \in T$ and $b := T(i, j - 1) \succ T(i - 1, j)$.

Then

Theorem 5.7. *For any skew diagram λ/μ of $l(\lambda) = 3$, equality (5.5) holds.*

Proof. In this proof, we use some maps which are defined in detail in appendix B. For a summary of this proof, see the maps and their relations in the following diagram:

$$
\begin{array}{ccccccc}
P_2^{\times} & \sqcup & & P_2^{\circ} & & = & P_2 \\
\Big\downarrow g & & {\scriptstyle f_2^{23}}\swarrow & & \searrow{\scriptstyle f_2^{13}} & & \\
& & P_1^{12} & \sqcup & P_1^{23} & = & P_1 \\
& & {\scriptstyle f_1^{12}}\searrow & & \swarrow{\scriptstyle f_1^{23}} & & \\
\operatorname{Im} g & \sqcup & & P_0 & & = & \tilde{P}
\end{array}
\qquad (5.10)
$$

Here, $P_2^{\times}$ denotes $P_2(C_n; \mu, \lambda)^{\times}$, for instance.

For λ/μ is a skew diagram of $l(\lambda) = 3$, there does not exist any $\mathbf{p} \in P(C_n; \mu, \lambda)$ which possesses more than two transposed pairs of paths. Therefore, we have $P(C_n; \mu, \lambda) = P_0(C_n; \mu, \lambda) \sqcup P_1(C_n; \mu, \lambda) \sqcup P_2(C_n; \mu, \lambda)$. Let $P_2(C_n; \mu, \lambda)^{\times}$ be the subset of $P_2(C_n; \mu, \lambda)$ (see figure 12) which consists of all $\mathbf{p} \in P_2(C_n; \mu, \lambda)$ such that the points $u' := u + (-1, 1)$ and $v' := v + (1, -1)$ are on $\mathbf{p} = (p_1, p_2, p_3)$, where u (resp. v) is the leftmost intersecting point of (p_1, p_3) (resp. the rightmost intersecting point of (p_2, p_3)). Let $P_1^{ij}(C_n; \mu, \lambda)$ $(1 \leqslant i < j \leqslant 3)$ be the set of all $\mathbf{p} = (p_1, p_2, p_3) \in P_1(C_n; \mu, \lambda)$ such that (p_i, p_j) is transposed. Let $P_2(C_n; \mu, \lambda)^{\circ} := P_2(C_n; \mu, \lambda) \setminus P_2(C_n; \mu, \lambda)^{\times}$. Let

$$
\begin{aligned}
f_2^{ij} &: P_2(C_n; \mu, \lambda)^{\circ} \to P_1(C_n; \mu, \lambda), \qquad (i, j) = (1, 3), (2, 3), \\
f_1^{ij} &: P_1^{ij}(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda), \qquad (i, j) = (1, 2), (2, 3)
\end{aligned}
$$

be the maps that resolve the transposed pair (p_i, p_j) in $\mathbf{p} = (p_1, p_2, p_3) \in P_k(C_n; \mu, \lambda)$, which are defined in section B.2 (see also (5.10)). These maps are weight-preserving injections (lemmas B.4 and B.5). We remark that the set $P_2(C_n; \mu, \lambda)^{\circ}$ consists of all $\mathbf{p}$ in $P_2(C_n; \mu, \lambda)$ such that f_2^{13} or f_2^{23} is well defined (in fact, both of them are well defined), while $P_2(C_n; \mu, \lambda)^{\times}$ consists of all $\mathbf{p} \in P_2(C_n; \mu, \lambda)$ such that both f_2^{13} and f_2^{23} are not well defined.

By lemma B.6, we have $\operatorname{Im} f_1^{12} \cap \operatorname{Im} f_1^{23} = \operatorname{Im}\big(f_1^{23} \circ f_2^{13}\big) = \operatorname{Im}\big(f_1^{12} \circ f_2^{23}\big)$, and therefore,

$$
-\sum_{\mathbf{p} \in P_1(C_n;\mu,\lambda)} z_a^{\mathbf{p}} + \sum_{\mathbf{p} \in P_2(C_n;\mu,\lambda)^{\circ}} z_a^{\mathbf{p}} = -\sum_{\mathbf{p} \in \operatorname{Im} f_1^{12} \cup \operatorname{Im} f_1^{23}} z_a^{\mathbf{p}}. \qquad (5.11)
$$

Let $g : P_2(C_n; \mu, \lambda)^{\times} \to \tilde{P}(C_n; \mu, \lambda)$ be the weight-preserving injection defined in section B.2 (see also figure 12). By lemma B.3 (1), we have

$$
\sum_{\mathbf{p} \in P_0(C_n;\mu,\lambda) \sqcup P_2(C_n;\mu,\lambda)^{\times}} z_a^{\mathbf{p}} = \sum_{\mathbf{p} \in \tilde{P}(C_n;\mu,\lambda)} z_a^{\mathbf{p}}. \qquad (5.12)
$$

Combining (5.11) and (5.12), we obtain

$$
\begin{aligned}
\chi_{\lambda/\mu,a} &= \sum_{\mathbf{p} \in \tilde{P}(C_n;\mu,\lambda)} z_a^{\mathbf{p}} - \sum_{\mathbf{p} \in \operatorname{Im} f_1^{12} \cup \operatorname{Im} f_1^{23}} z_a^{\mathbf{p}} \\
&= \sum_{T \in \widetilde{\mathrm{Tab}}(C_n,\lambda/\mu)} z_a^{T} - \sum_{\mathbf{p} \in \operatorname{Im} f_1^{12} \cup \operatorname{Im} f_1^{23}} z_a^{T(\mathbf{p})}.
\end{aligned}
$$

By lemma B.7, the set $\big\{T(\mathbf{p}) \mid \mathbf{p} \in \operatorname{Im} f_1^{12} \cup \operatorname{Im} f_1^{23}\big\}$ consists of all the tableaux in $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$ prohibited by the extra rules (**E-2R**) and (**E-3R**). □

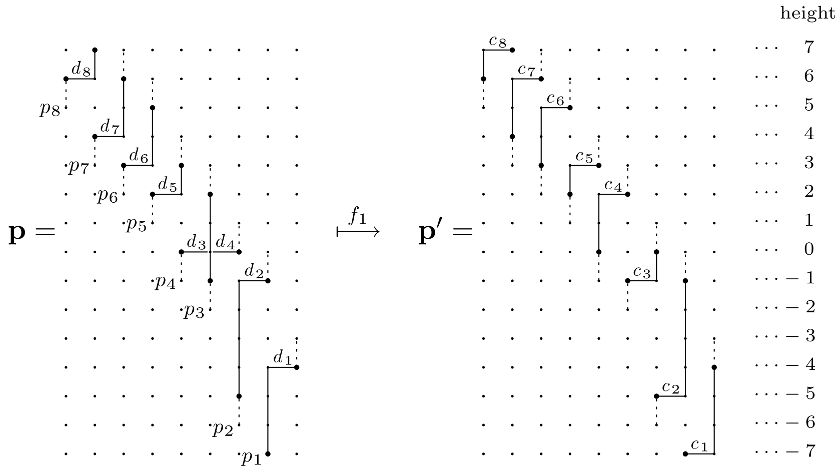


Figure 8. An example of $\mathbf{p} \in P_1(C_n; \mu, \lambda)$ for one-column λ/μ. For this $\mathbf{p}$, the map f_1 in (5.14) is given by $f_1 = r_6^{18} \circ r_5^{17} \circ r_4^{16} \circ r_4^{27} \circ r_3^{26} \circ r_2^{25} \circ r_1^{24} \circ r_0^{34}$. The tableau $T(\mathbf{p}')$ does not satisfy (**E-1C**).

5.4. Skew diagrams of at most two columns

In this subsection, we conjecture the tableaux description for skew diagrams λ/μ of at most two columns and prove it for $l(\lambda) \leqslant 4$. We assume that $l(\lambda) \leqslant n+1$.

The case of one-column. Let λ/μ be a skew diagram of one column (i.e., $l(\lambda') = 1$). Let $\mathrm{Tab}(C_n, \lambda/\mu)$ be the set of all the HV-tableaux T (actually, the horizontal rule (**H**) is not required) with the following condition:

(**E-1C**) If T contains a subtableau

$$\begin{array}{|c|}\hline c_1 \\ \hline \vdots \\ \hline c_l \\ \hline \end{array} \tag{5.13}$$

such that $l \geqslant 2$, $c_1 = c$ and $c_l = \overline{c}$ for some $1 \leqslant c \leqslant n$, then $l - 1 \leqslant n - c$.

The following theorem is due to [22]. We reproduce it using the paths description.

Theorem 5.8 ([22]). *For any skew diagram λ/μ of $l(\lambda') = 1$ and $l(\lambda) \leqslant n+1$, equality (5.5) holds.*

Proof. By $l(\lambda') = 1$, there does not exist any $\mathbf{p} \in P(C_n; \mu, \lambda)$ which contains more than one transposing pair of paths, and therefore, we have $P(C_n; \mu, \lambda) = P_0(C_n; \mu, \lambda) \sqcup P_1(C_n; \mu, \lambda)$. We can define a weight-preserving, sign-inverting injection (which is well defined if $l(\lambda) \leqslant n+1$)

$$f_1 : P_1(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda), \tag{5.14}$$

using the maps r_y^{ij} in B.1 (see also figure 8), and show that the set $\{T(\mathbf{p}) \mid \mathbf{p} \in \mathrm{Im} f_1\}$ consists of all the tableaux in $\widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$ prohibited by the (**E-1C**) rule. □

The case of two-column. Let λ/μ be a skew diagram of two columns, i.e., $l(\lambda') = 2$. Let $T \in \widetilde{\mathrm{Tab}}(C_n, \lambda/\mu)$ be a tableau that contains a subtableau

$$T' = \begin{array}{|c|}\hline c_1 \\ \hline \vdots \\ \hline c_l \\ \hline \end{array} \subset T \tag{5.15}$$

such that $l \geqslant 2$, $c_1 = n+2-l$, $c_l = \overline{n+2-l}$ and every proper subtableau of T' satisfies (**E-1C**). Let $\tilde{\lambda}/\tilde{\mu}$ be the one-column shape of T'. Then we can pick $\mathbf{p} \in P_0(C_n; \tilde{\mu}, \tilde{\lambda})$ such that $T(\mathbf{p}) = T'$. For T' does not satisfy the extra rule (**E-1C**), we have $\mathbf{p} \in \mathrm{Im}\, f_1$, where f_1 is injection (5.14) in the proof of theorem 5.8. Let $f_1^{-1}(\mathbf{p}) = (p_1, \ldots, p_l) \in P_1(C_n; \tilde{\mu}, \tilde{\lambda})$ be the inverse image of $\mathbf{p}$. Then set (see figure 8)

$$d_i = d_i(T') := \begin{cases} L_a^1(s^i), & i = 1, \ldots, l, \qquad i \neq k, k+1, \\ \overline{n}, & i = k, \\ n, & i = k+1, \end{cases} \tag{5.16}$$

where $L_a^1(s^i)$ is the h-label (defined in (5.1)) of the unique eastward step s^i in p_i, and k is the number such that $c_k \preceq n$ and $c_{k+1} \succeq \overline{n}$. Then, one can show that

$$\{c_1, \ldots, c_k\} \cup \{\overline{d}_{k+2}, \ldots, \overline{d}_l\} = \{n, n-1, \ldots, n+2-l\}, \tag{5.17}$$

$$\{\overline{d}_1, \ldots, \overline{d}_{k-1}\} \cup \{c_{k+1}, \ldots, c_l\} = \{\overline{n}, \overline{n-1}, \ldots, \overline{n+2-l}\}. \tag{5.18}$$

For example, these elements for all T' as in (5.15) of $l \leqslant 4$ are given in table 1.

Now we define $\mathrm{Tab}(C_n, \lambda/\mu)$ as the set of all the HV-tableaux T with the following condition: (**E-2C**) Let T' be any subtableau of T (excluding $a_1, \ldots, a_k, b_{k+1}, \ldots, b_l$)

$$T' = \begin{array}{c|c|c} & c_1 & a_1 \\ \cline{2-2} & \vdots & \vdots \\ \cline{2-2} & c_k & a_k \\ \cline{2-2} b_{k+1} & c_{k+1} & \\ \cline{2-2} \vdots & \vdots & \\ \cline{2-2} b_l & c_l & \\ \end{array} \subset T \tag{5.19}$$

such that $l \geqslant 2$, $c_1 = n+2-l$, $c_l = \overline{n+2-l}$, $c_k \preceq n$, $c_{k+1} \succeq \overline{n}$, and every proper subtableau in T' satisfies the extra condition (**E-1C**). Let (i_1, j_1) be the position of the top of the subtableau T' in (5.19) (i.e., the position of c_1). Then one of the following conditions holds:

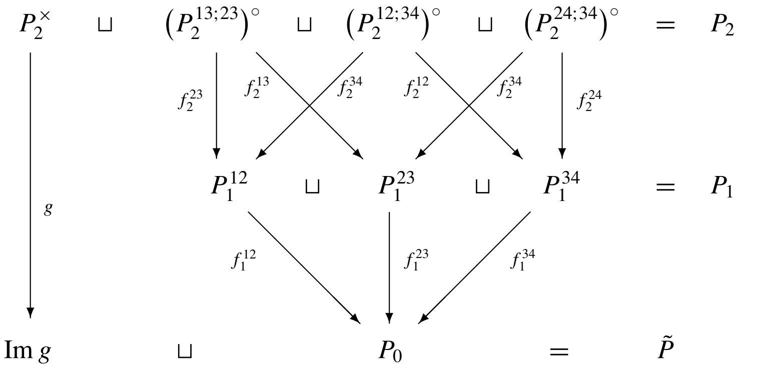
(1) $(i_1 + i - 1, j_1 - 1) \in \lambda/\mu$ and $a_i := T(i_1 + i - 1, j_1 + 1) \prec d_i(T')$ for some $1 \leqslant i \leqslant k$.
(2) $(i_1 + i - 1, j_1 + 1) \in \lambda/\mu$ and $b_i := T(i_1 + i - 1, j_1 - 1) \succ d_i(T')$ for some $k+1 \leqslant i \leqslant l$.

We remark that the extra rule (**E-2C**) is reduced to the extra rule (**E-1C**), if $l(\lambda') = 1$. We conjecture that

Conjecture 5.9. *For any skew diagram λ/μ of $l(\lambda') = 2$ and $l(\lambda) \leqslant n+1$, equality (5.5) holds.*

Theorem 5.10. *Conjecture 5.9 is true for $l(\lambda) \leqslant 4$.*

Proof. If $l(\lambda) \leqslant 3$, then the extra rule (**E-2C**) for $l(\lambda) \leqslant 3$ coincides with the extra rule (**E-2R**) with (**E-3R**). For a summary of the proof for $l(\lambda) = 4$, which is parallel to that of theorem 5.7, see the maps and their relations in the following diagram:



(5.20)

Here, $(P_2^{13;23})^\circ$ denotes $P_2^{13;23}(C_n;\mu,\lambda)^\circ$, for instance.

There does not exist $\mathbf{p} \in P(C_n;\mu,\lambda)$ that contains more than two transposed pairs of paths, and therefore,

$$P(C_n;\mu,\lambda) = P_0(C_n;\mu,\lambda) \sqcup P_1(C_n;\mu,\lambda) \sqcup P_2(C_n;\mu,\lambda).$$

As in the proof of theorem 5.7, we define $P_1^{ij}(C_n;\mu,\lambda)$ $(1 \leqslant i < j \leqslant 4)$ as the set of all $\mathbf{p} = (p_1,\ldots,p_4) \in P_1(C_n;\mu,\lambda)$ such that (p_i,p_j) is transposed. Then we have

$$P_1(C_n;\mu,\lambda) = P_1^{12}(C_n;\mu,\lambda) \sqcup P_1^{23}(C_n;\mu,\lambda) \sqcup P_1^{34}(C_n;\mu,\lambda).$$

Similarly, we define $P_2^{ij;km}(C_n;\mu,\lambda)$ $(1 \leqslant i < j \leqslant 4, 1 \leqslant k < m \leqslant 4)$ as the set of all $\mathbf{p} = (p_1,\ldots,p_4) \in P_2(C_n;\mu,\lambda)$ such that (p_i,p_j) and (p_k,p_m) are transposed. Then we have

$$P_2(C_n;\mu,\lambda) = P_2^{13;23}(C_n;\mu,\lambda) \sqcup P_2^{12;34}(C_n;\mu,\lambda) \sqcup P_2^{24;34}(C_n;\mu,\lambda).$$

Let $P_2^{13;23}(C_n;\mu,\lambda)^\times$ be the set that consists of all $\mathbf{p} = (p_1,\ldots,p_4) \in P_2^{13;23}(C_n;\mu,\lambda)$ which satisfy one of the following conditions, where u (resp. $v = u - (2,0)$) is the unique intersecting point of (p_1,p_3) (resp. (p_2,p_3)) at height 0:

(1) Both points $u + (-1,1)$ and $v + (1,-1)$ are on $\mathbf{p}$.
(2) All four points $u + (-1,1)$, $v + (1,-2)$, $v + (-1,1)$ and $v + (0,2)$ are on $\mathbf{p}$.

Let $P_2^{24;34}(C_n;\mu,\lambda)^\times$ be the set of all $\mathbf{p} \in P_2^{24;34}(C_n;\mu,\lambda)$ such that $\omega(\mathbf{p}) \in P_2^{13;23}(C_n;\tilde{\mu},\tilde{\lambda})^\times$, where ω is a map that rotates $\mathbf{p}$ by 180° defined as in (B.3). Let $P_2^{12;34}(C_n;\mu,\lambda)^\times$ be the set that consists of all $\mathbf{p} = (p_1,\ldots,p_4) \in P_2^{12;34}(C_n;\mu,\lambda)$ such that all four points $u + (-1,1)$, $u + (-2,2)$, $w + (1,-1)$ and $w + (2,-2)$ are on $\mathbf{p}$, where u (resp. $w = u - (3,0)$) is the unique intersecting point of (p_1,p_2) (resp. (p_3,p_4)) at height 0. Let $P_2^{13;23}(C_n;\mu,\lambda)^\circ := P_2^{13;23}(C_n;\mu,\lambda) \setminus P_2^{13;23}(C_n;\mu,\lambda)^\times$, etc. We can define a weight-preserving, sign-inverting injection

$$f_2^{ij} : P_2^{km;k'm'}(C_n;\mu,\lambda)^\circ \to P_1(C_n;\mu,\lambda), \qquad (i,j) = (k,m),(k',m'),$$
$$f_1^{ij} : P_1^{ij}(C_n;\mu,\lambda) \to P_0(C_n;\mu,\lambda),$$

which resolve the transposed pair (p_i,p_j) of $\mathbf{p} = (p_1,\ldots,p_4) \in P_k(C_n;\mu,\lambda)$ as the maps f_2^{ij}, f_1^{ij} in the proof of theorem 5.7. These maps can also be defined as the composition of the maps r_y^{ij} given in section B.1. We remark that $P_2^{km;k'm'}(C_n;\mu,\lambda)^\circ$ consists of all $\mathbf{p} \in P_2^{km;k'm'}(C_n;\mu,\lambda)$ such that f_2^{ij} for some $1 \leqslant i < j \leqslant 4$ is well defined (in fact, all f_2^{ij}

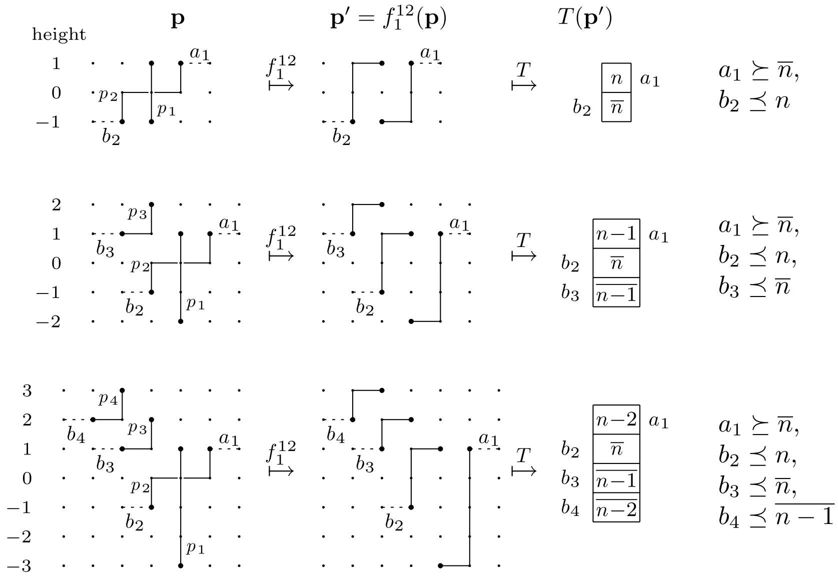


Figure 9. Examples of $\mathbf{p} = (p_1, \ldots, p_4) \in P_1^{12}(C_n; \mu, \lambda)$, the map $f_1^{12} : P_1^{12}(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda)$ and the subtableaux of $T(f_1^{23}(\mathbf{p}))$. If the (E) step for a_i (resp. b_i) exists, then a_i (resp. b_i) satisfies the condition as above, which implies the corresponding tableau is prohibited by (**E-2C**).

Table 1. The table of $(d_1, \ldots, d_l)$ for one-column tableaux T' as in (5.15) of $l \leqslant 4$.

T'	$n, \bar{n}$	$n-1, n, \overline{n-1}$	$n-1, \bar{n}, \overline{n-1}$	$n-2, n-1, n, \overline{n-2}$	$n-2, n-1, \bar{n}, \overline{n-2}$	$n-2, n-1, \overline{n-1}, \overline{n-2}$	$n-2, n, \overline{n-1}, \overline{n-2}$	$n-2, \bar{n}, \overline{n-1}, \overline{n-2}$
d_1	$\bar{n}$	n	$\bar{n}$	$n-1$	$n-1$	n	n	$\bar{n}$
d_2	n	$\bar{n}$	n	n	$\bar{n}$	$\bar{n}$	$\bar{n}$	n
d_3		n	$\bar{n}$	$\bar{n}$	n	n	n	$\bar{n}$
d_4				n	$\bar{n}$	$\bar{n}$	$\overline{n-1}$	$\overline{n-1}$

are well defined), while $P_2^{km;k'm'}(C_n; \mu, \lambda)^\times$ consists of all $\mathbf{p} \in P_2^{km;k'm'}(C_n; \mu, \lambda)$ such that f_2^{ij} for any $1 \leqslant i < j \leqslant 4$ is not well defined. These maps satisfy

$$\begin{aligned}
&\mathrm{Im}\big(f_1^{34} \circ f_2^{12}\big) = \mathrm{Im}\big(f_1^{12} \circ f_2^{34}\big) = \mathrm{Im}\, f_1^{12} \cap \mathrm{Im}\, f_1^{34},\\
&\mathrm{Im}\big(f_1^{23} \circ f_2^{13}\big) = \mathrm{Im}\big(f_1^{12} \circ f_2^{23}\big) = \mathrm{Im}\, f_1^{12} \cap \mathrm{Im}\, f_1^{23},\\
&\mathrm{Im}\big(f_1^{34} \circ f_2^{24}\big) = \mathrm{Im}\big(f_1^{23} \circ f_2^{34}\big) = \mathrm{Im}\, f_1^{23} \cap \mathrm{Im}\, f_1^{34},\\
&\mathrm{Im}\, f_1^{12} \cap \mathrm{Im}\, f_1^{23} \cap \mathrm{Im}\, f_1^{34} = \phi,
\end{aligned}$$

which can be proved by using the forms of the subtableaux in $T(\mathbf{p})$ of $\mathbf{p} \in \mathrm{Im} f_1^{ij}$ for $(i, j) = (1, 2), (2, 3), (3, 4)$ (see table 1 and figure 9). We can also define a weight-preserving, sign-preserving injection

$$g : P_2(C_n; \mu, \lambda)^\times \to \tilde{P}(C_n; \mu, \lambda)$$

on $P_2(C_n;\mu,\lambda)^\times := P_2^{13;23}(C_n;\mu,\lambda)^\times \sqcup P_2^{12;34}(C_n;\mu,\lambda)^\times \sqcup P_2^{24;34}(C_n;\mu,\lambda)^\times$, which satisfies $\mathrm{Im}\, g \sqcup P_0(C_n;\mu,\lambda) = \tilde{P}(C_n;\mu,\lambda)$, as in the proof of theorem 5.7. Then we similarly obtain equality (5.5) by the following lemma. □

Lemma 5.11. *Let λ/μ be a skew diagram of $l(\lambda') \leqslant 2$ and $l(\lambda) = n+1$. For $\mathbf{p} \in \tilde{P}(C_n;\mu,\lambda)$, $\mathbf{p} \in \mathrm{Im}\, f_1^{12} \cup \mathrm{Im}\, f_1^{23} \cup \mathrm{Im}\, f_1^{34}$ if and only if $T(\mathbf{p})$ is prohibited by* (**E-2C**). *(See figure 9 for example.)*

Acknowledgment

We thank T Arakawa and S Okada for useful discussions.

Appendix A. Classical projection of $\chi_{\lambda,a}$

In this section, we give a 'classical projection' of the determinant $\chi_{\lambda,a}$ in (2.23), the one obtained by dropping the spectral parameters $a \in \mathbb{C}$. We prove that the classical projection of $\chi_{\lambda,a}$ coincides with the character for the representation of $U_q(\mathfrak{g})$ defined in [9].

Let $\beta : \mathbb{Z}\big[Y_{i,a}^{\pm 1}\big]_{i=1,\ldots,n;a\in\mathbb{C}} \to \mathbb{Z}\big[y_i^{\pm 1}\big]_{i=1,\ldots,n}$ be the classical projection, i.e., the algebra homomorphism defined by $\beta(Y_{i,a}) = y_i$. Identifying $\mathcal{Z}$ with $\mathcal{Y}$ by the isomorphism in the proof of proposition 2.1, $\beta|_{\mathcal{Z}}$ is the map $\mathcal{Z} \to \mathbb{Z}\big[z_i^{\pm 1}\big]_{i=1,\ldots,N}$ such that $\beta(z_{i,a}) = z_i$, $\beta(z_{0,a}) = 1$ (for B_n) and $\beta(z_{\bar{i},a}) = z_i^{-1}$ (for B_n, C_n and D_n). The homomorphism β sends the q-character $\chi_q(V)$ for any finite-dimensional representation V of $U_q(\hat{\mathfrak{g}})$ to the character $\chi(V)$ of $U_q(\mathfrak{g})$ for V as a $U_q(\mathfrak{g})$-module [15].

Let $\chi_{\lambda,a}$ be determinant (2.23) with $\mu = \phi$. Let $\chi_\lambda \in \mathbb{Z}\big[z_i^{\pm 1}\big]_{i=1,\ldots,N}$ be the character of $\mathfrak{g}$ for the irreducible representation with highest weight λ. For any partitions μ, ν, λ, let $c_{\mu\nu}^{\lambda}$ be the Littlewood–Richardson coefficient [25]. Then we have

Theorem A.1. *For any λ such that $l(\lambda) \leqslant n$,*

$$\beta(\chi_{\lambda,a}) = \begin{cases} \chi_\lambda, & \text{if } \mathfrak{g} \text{ is of type } A_n, \\ \sum_{\kappa,\mu} c_{(2\kappa)',\mu}^{\lambda}\chi_\mu, & B_n, \\ \sum_{\kappa,\mu} c_{2\kappa,\mu}^{\lambda}\chi_\mu, & C_n, \\ \sum_{\kappa,\mu} c_{(2\kappa)',\mu}^{\lambda}\tilde{\chi}_\mu, & D_n, \end{cases} \tag{A.1}$$

where

$$\tilde{\chi}_\lambda := \begin{cases} \chi_\lambda, & \text{if} \quad 1 \leqslant l(\lambda) \leqslant n-1, \\ \chi_\lambda + \chi_{\sigma(\lambda)}, & \text{if} \quad l(\lambda) = n, \end{cases}$$

for D_n, where σ is induced from the automorphism of the Dynkin diagram.

Proof. Let Λ be the graded ring of symmetric functions with countable many variables $z_1, z_2, \ldots,$ and let $S_\lambda \in \Lambda$ be the *Schur function*. It is well known that S_λ satisfies the Jacobi–Trudi identity

$$S_\lambda = \det\big(h_{\lambda_i - i + j}\big)_{1\leqslant i,j\leqslant l(\lambda)} = \det\big(e_{\lambda'_i - i + j}\big)_{1\leqslant i,j\leqslant l(\lambda')},$$

where $h_i, e_i \in \Lambda$ are defined as

$$\prod_{k=1}^{\infty}(1 - z_k x)^{-1} = \sum_{i=0}^{\infty} h_i x^i, \qquad \prod_{k=1}^{\infty}(1 + z_k x) = \sum_{i=0}^{\infty} e_i x^i.$$

Let $\varphi : \Lambda \to \Lambda$ be the algebra automorphism defined by

$$\varphi(e_{\rm i}) = e_{\rm i} - e_{{\rm i}-2}, \qquad \varphi^{-1}(e_{\rm i}) = \sum_{m=0}^{\infty} e_{{\rm i}-2m},$$

$$\varphi(h_i) = \sum_{m=0}^{\infty} h_{i-2m}, \qquad \varphi^{-1}(h_i) = h_i - h_{i-2}.$$

(All the four conditions are equivalent to each other.)

Set $\Lambda_n := \mathbb{Z}[z_1, \ldots, z_n]^{\mathfrak{S}_n}$. Let $\pi_{n+1} : \Lambda \to \Lambda_{n+1}/\langle z_1 \cdots z_{n+1} - 1\rangle$ be the map induced from the natural projection $\Lambda \to \Lambda_{n+1}$ and let $\pi_{Sp(2n)}$ and $\pi_{O(N)}$ be the *specialization homomorphisms* in [23]. We define ρ_n as

$$\rho_n = \begin{cases} \pi_{n+1}, & (A_n) \\ \pi_{O(2n+1)} \circ \varphi^{-1}, & (B_n) \\ \pi_{Sp(2n)} \circ \varphi, & (C_n) \\ \pi_{O(2n)} \circ \varphi^{-1}. & (D_n) \end{cases}$$

By the properties of $\pi_{O(N)}$ and $\pi_{Sp(2n)}$ [23] and the definitions of $h_{i,a}$ and $e_{i,a}$ in (2.21) and (2.20), we have $\beta(\chi_{\lambda,a}) = \rho_n(S_\lambda)$ for any Young diagram λ of $l(\lambda) \leqslant n$. Therefore, for A_n, (A.1) is obvious by the fact that $\pi_{n+1}(S_\lambda) = \chi_\lambda$, while for B_n, C_n and D_n, (A.1) are obtained by the equalities [23, 30]

$$\prod_{i,j\geqslant 1} \frac{1}{(1 - z_i\tilde{z}_j)} = \sum_\lambda S_\lambda(z) S_\lambda(\tilde{z}),$$

$$\frac{\prod_{1\leqslant i\leqslant j}(1 - \tilde{z}_i\tilde{z}_j)}{\prod_{i,j\geqslant 1}(1 - z_i\tilde{z}_j)} = \sum_\lambda \chi_O(\lambda)(z) S_\lambda(\tilde{z}),$$

$$\frac{\prod_{1\leqslant i< j}(1 - \tilde{z}_i\tilde{z}_j)}{\prod_{i,j\geqslant 1}(1 - z_i\tilde{z}_j)} = \sum_\lambda \chi_{Sp}(\lambda)(z) S_\lambda(\tilde{z}),$$

where $\chi_{Sp}(\lambda), \chi_O(\lambda) \in \Lambda$ are the *universal character* of Sp and O [23], and the Littlewood's lemma [24]

$$\prod_{1\leqslant i\leqslant j} \frac{1}{(1 - z_i z_j)} = \sum_\kappa S_{2\kappa}(z), \qquad \prod_{1\leqslant i< j} \frac{1}{(1 - z_i z_j)} = \sum_\kappa S_{(2\kappa)'}(z).$$ □

Remark A.2. The right-hand side of (A.1) is the character of the representation $W_G(\lambda)$ defined in [9]. Therefore, by theorem A.1, under the classical projection, conjecture 2.2 reduces to conjecture 2 in [9] of the existence of an irreducible representation of $U_q(\hat{\mathfrak{g}})$, which is proved by [8] for $\lambda = (i^m)$ such that $m \geqslant 1$ and $1 \leqslant i \leqslant n$ (A_n and B_n), $1 \leqslant i \leqslant n-1$ (C_n), $1 \leqslant i \leqslant n-2$ (D_n).

Appendix B. The weight-preserving maps for C_n case

In this section, we define some weight-preserving maps and give their properties which we use in the proof of theorems 5.6 and 5.7.

B.1. The map r_y

In this subsection, we give weight-preserving maps for a pair of h-paths of type C_n. These maps are used to define the maps in section B.2. First, we define the map r_y for $y = 0, 1, \ldots, n-1$, which is defined on all $(p_1, p_2) \in P(C_n) \times P(C_n)$ that satisfy certain condition ($\mathbf{R}_y$).

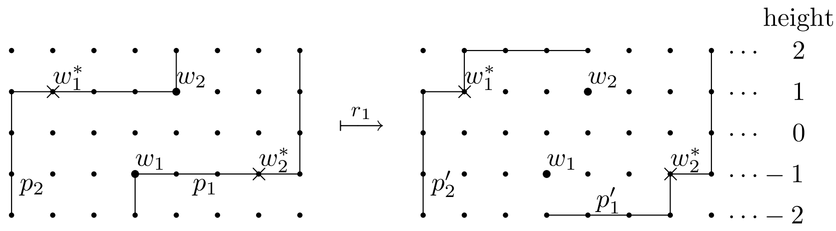


Figure 10. An example of two paths and the map r_y for $y = 1$.

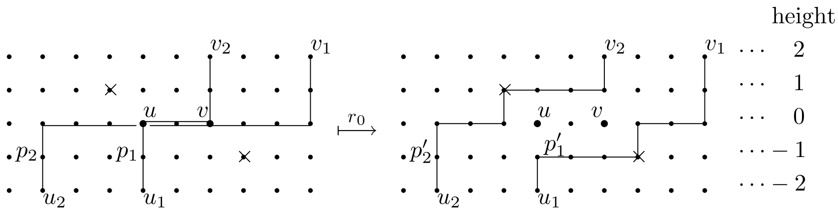


Figure 11. An example of specially intersecting, transposed paths and the map r_0.

Set $(x, y) \pm (x', y') := (x \pm x', y \pm y')$.

Let $y = 1, \ldots, n-1$. For any $p_1, p_2 \in P(C_n)$, let w_1 (resp. w_2) be the leftmost point of height $-y$ on p_1 (resp. the rightmost point of height y on p_2), i.e., if $w_1 = (x_1, -y)$ and $w_2 = (x_2, y)$, then

$$x_1 = \min\{x \mid (x, -y) \text{ is on } p_1\}, \qquad x_2 = \max\{x \mid (x, y) \text{ is on } p_2\}.$$

See figure 10 for example. Note that $w_1 - (0, 1)$ is on p_1 and $w_2 + (0, 1)$ is on p_2. We define the condition $(\mathbf{R}_y)$ for any $p_1, p_2 \in P(C_n)$ as follows:

$(\mathbf{R}_y)$ $w_1^* := w_1 + (-y-1, 2y)$ is on p_2 and $w_2^* := w_2 + (y+1, -2y)$ is on p_1.

For any $p_1, p_2 \in P(C_n)$ which satisfy $(\mathbf{R}_y)$, we define $r_y(p_1, p_2) = (p_1', p_2')$ $(y = 1, \ldots, n-1)$ as (see figure 10)

$$\begin{aligned} p_1' &: u_1 \xrightarrow{p_1} w_1 - (0,1) \longrightarrow w_2^* - (0,1) \longrightarrow w_2^* \xrightarrow{p_1} v_1, \\ p_2' &: u_2 \xrightarrow{p_2} w_1^* \longrightarrow w_1^* + (0,1) \longrightarrow w_2 + (0,1) \xrightarrow{p_2} v_2. \end{aligned} \tag{B.1}$$

For the $y = 0$ case, let $p_1, p_2 \in P(C_n)$ satisfy the following condition:

$(\mathbf{R}_0)$ (p_1, p_2) is specially intersecting at height 0.

Then we define $r_0(p_1, p_2) = (p_1', p_2')$ as follows: if p_1 and p_2 are not transposed, then let w_1 (resp. w_2) be the leftmost point of height 0 on p_1 (resp. the rightmost point of height 0 on p_2), and set w_1^* and w_2^* as in $(\mathbf{R}_y)$ by putting $y = 0$. Then set (p_1', p_2') as in (B.1). If (p_1, p_2) is transposed (see figure 11 for example), then let u (resp. v) be the leftmost (resp. rightmost) intersecting point of p_1 and p_2 at height 0. We assume that $u - (0, 1)$ and $v + (0, 1)$ is on p_1 while $u - (1, 0)$ and $v + (1, 0)$ is on p_2. Set $r_0(p_1, p_2) = (p_1', p_2')$ by

$$\begin{aligned} p_1' &: u_1 \xrightarrow{p_1} u - (0,1) \longrightarrow v + (1,-1) \longrightarrow v + (1,0) \xrightarrow{p_2} v_1, \\ p_2' &: u_2 \xrightarrow{p_2} u - (1,0) \longrightarrow u + (-1,1) \longrightarrow v + (0,1) \xrightarrow{p_1} v_2. \end{aligned}$$

(Roughly speaking, r_0 'resolves' the transposed pair (p_1, p_2).) By (2.4) and the definition of the h-label of type C_n, we have

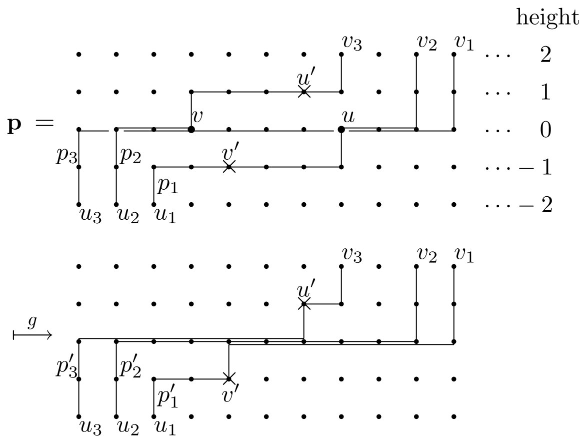


Figure 12. An example of $\mathbf{p} \in P_2(C_n; \mu, \lambda)^\times$ and the map g.

Lemma B.1. r_y $(y = 0, \ldots, n-1)$ *preserves the weight of* (p_1, p_2).

Let $1 \leqslant i < j \leqslant l$. For any $\mathbf{p} = (p_1, \ldots, p_l)$ such that the pair (p_i, p_j) satisfies $(\mathbf{R}_y)$ $(0 \leqslant y \leqslant n-1)$, we define $r_y^{ij}(\mathbf{p}) = (p'_1, \ldots, p'_l)$ by

$$(p'_i, p'_j) := r_y(p_i, p_j), \qquad p'_k := p_k, \quad (k \neq i, j). \tag{B.2}$$

By lemma B.1, it is obvious that

Proposition B.2. r_y^{ij} *preserves the weight for any* $0 \leqslant y \leqslant n-1$ *and* $1 \leqslant i < j \leqslant l$.

Remark that $r_y^{ij}(\mathbf{p})$ for $\mathbf{p} \in P(C_n; \mu, \lambda)$ may include an ordinarily intersecting pair of paths, which implies that $r_y^{ij}(\mathbf{p})$ is not necessarily an element of $P(C_n; \mu, \lambda)$.

B.2. The maps in the proof of theorem 5.7

In this subsection, λ/μ is a skew diagram of $l(\lambda) = 3$. In this case, we have $P(C_n; \mu, \lambda) = P_0(C_n; \mu, \lambda) \sqcup P_1(C_n; \mu, \lambda) \sqcup P_2(C_n; \mu, \lambda)$. We define some maps which we use in section 5.3 and show their properties.

The map g. For any $\mathbf{p} = (p_1, p_2, p_3) \in P_2(C_n; \mu, \lambda)$, let u be the leftmost intersecting point of p_1 and p_3, and let v be the rightmost intersecting point of p_2 and p_3 (see figure 12). Then set $u' := u + (-1, 1)$ and $v' := v + (1, -1)$. Let $P_2(C_n; \mu, \lambda)^\times$ be the set of all $\mathbf{p} \in P_2(C_n; \mu, \lambda)$ such that both u' and v' are on some p_i (actually, u' is on p_2 and v' is on p_1). For example, $\mathbf{p}$ in figure 12 is an element of $P_2(C_n; \mu, \lambda)^\times$. Let $\tilde{P}(C_n; \mu, \lambda)$ be the subset of $\mathfrak{P}(C_n; \mathbf{u}_\mu, \mathbf{v}_\lambda)$ defined in section 5.2. We define a map

$$g : P_2(C_n; \mu, \lambda)^\times \to \tilde{P}(C_n; \mu, \lambda)$$

as follows: for $\mathbf{p} = (p_1, p_2, p_3) \in P_2(C_n; \mu, \lambda)^\times$, set $g(\mathbf{p}) = (p'_1, p'_2, p'_3)$ as (see figure 12)

$$p'_1 : u_1 \xrightarrow{p_1} v' \longrightarrow v' + (0, 1) \xrightarrow{p_3} v_1,$$
$$p'_2 : u_2 \xrightarrow{p_2} v \longrightarrow u \xrightarrow{p_1} v_2,$$
$$p'_3 : u_3 \xrightarrow{p_3} u' - (0, 1) \longrightarrow u' \xrightarrow{p_2} v_3.$$

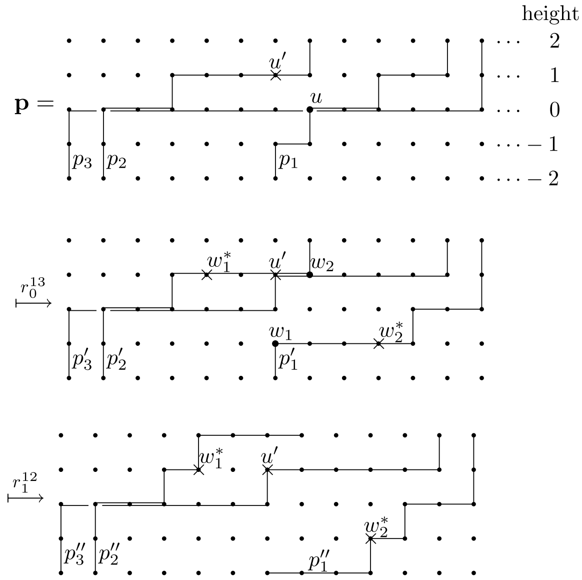


Figure 13. An example of $\mathbf{p} \in P_2(C_n; \mu, \lambda)^\circ$ and the map f_2^{13} of case (f_2^{13}-b).

Then we easily show that

Lemma B.3.

(1) g *is a weight-preserving, sign-preserving injection and* $\tilde{P}(C_n; \mu, \lambda) = \operatorname{Im} g \sqcup P_0(C_n; \mu, \lambda)$.

(2)

$$\left\{ T(\mathbf{p}) \mid \mathbf{p} \in \operatorname{Im} g \right\} = \left\{ T \in \widetilde{\operatorname{Tab}}(C_n; \lambda/\mu) \mid T \textit{ contains } \begin{array}{|c|c|}\hline \bar{n} & n \\ \hline \bar{n} & n \\ \hline \bar{n} & n \\ \hline \end{array} \right\}.$$

The maps f_2^{13} and f_2^{23}. Let $P_1^{ij}(C_n; \mu, \lambda)$ $(1 \leqslant i < j \leqslant 3)$ be the set of all $\mathbf{p} = (p_1, p_2, p_3) \in P_1(C_n; \mu, \lambda)$ such that (p_i, p_j) is transposed. Then we have $P_1(C_n; \mu, \lambda) = P_1^{12}(C_n; \mu, \lambda) \sqcup P_1^{23}(C_n; \mu, \lambda)$. Let $P_2(C_n; \mu, \lambda)^\circ := P_2(C_n; \mu, \lambda) \setminus P_2(C_n; \mu, \lambda)^\times$. We define a map

$$f_2^{13} : P_2(C_n; \mu, \lambda)^\circ \to P_1^{23}(C_n; \mu, \lambda)$$

as follows, using the weight-preserving maps r_y^{ij} defined in (B.2) (roughly speaking, f_2^{13} resolves the transposed pair (p_1, p_3) of $\mathbf{p} = (p_1, p_2, p_3)$ without producing any ordinarily intersecting paths): for any $\mathbf{p} = (p_1, p_2, p_3) \in P_2(C_n; \mu, \lambda)^\circ$, let u be the leftmost intersecting point of p_1 and p_3 and $u' := u + (-1, 1)$. Set $\mathbf{p}' := (p_1', p_2', p_3') = r_0^{13}(\mathbf{p})$, which is well defined. Then

Case $\left(f_2^{13}\text{-a}\right)$. If u' is not on any p_i, set $f_2^{13}(\mathbf{p}) = \mathbf{p}'$, which is in $P_1^{23}(C_n; \mu, \lambda)$. (Otherwise, u' is on p_2 and (p_2', p_3') is ordinarily intersecting.)

Case $\left(f_2^{13}\text{-b}\right)$. Otherwise, (p_1', p_2') satisfies the condition $(\mathbf{R}_1)$ in section B.1. Set $f_2^{13}(\mathbf{p}) = r_1^{12}(\mathbf{p}')$, which is in $P_1^{23}(C_n; \mu, \lambda)$ (see figure 13).

We remark that if $\mathbf{p} \in P_2(C_n; \mu, \lambda)^\times$, then (p_2', p_3') is ordinarily intersecting, but the procedure of $\left(f_2^{13}\text{-b}\right)$ is not well defined, for (p_1', p_2') does not satisfy $(\mathbf{R}_1)$.

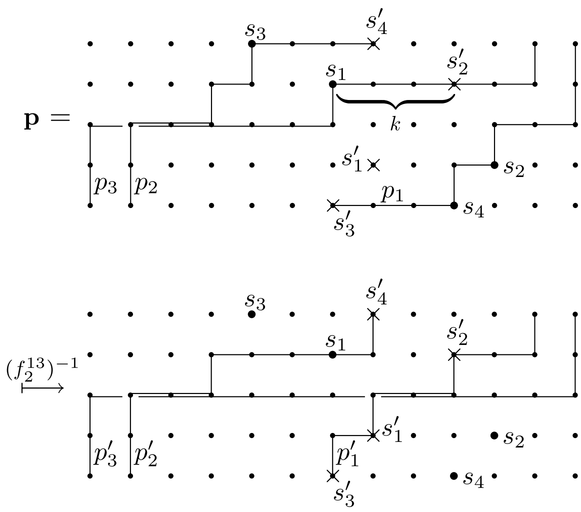


Figure 14. An example of $\mathbf{p} \in P_1^{23}(C_n; \mu, \lambda)$ which satisfy $(\mathbf{F_2^{13}\text{-}b})$, and the inverse procedure of $(f_2^{13}\text{-b})$.

We also define

$$f_2^{23} : P_2(C_n; \mu, \lambda)^\circ \to P_1^{12}(C_n; \mu, \lambda)$$

by $\omega \circ f_2^{13} \circ \omega$, where

$$\omega : \mathfrak{P}(\mathbf{u}_\mu, \mathbf{v}_\lambda) \to \mathfrak{P}(\mathbf{u}_{\tilde{\mu}}, \mathbf{v}_{\tilde{\lambda}}) \tag{B.3}$$

is a map that rotates $\mathbf{p}$ by 180° around a fixed point $(x, 0)$ such that $2x - \lambda_1 + l(\lambda) - 1 \in \mathbb{Z}_{\geqslant 0}$ (so that $\tilde{\lambda}$ and $\tilde{\mu}$ are partitions). Then, f_2^{23} resolves the transposed pair (p_2, p_3) of $\mathbf{p} = (p_1, p_2, p_3) \in P_2(C_n; \mu, \lambda)^\circ$.

Next, we give the conditions to describe the images of f_2^{13} and f_2^{23}. For any $\mathbf{p} \in P_1^{23}(C_n; \mu, \lambda)$ (see figure 14), let s_1 be the leftmost point of height 1 on p_3, s_2 be the rightmost point of height -1 on p_1, s_3 be the leftmost point of height 2 on p_2, and s_4 be the rightmost point of height -2 on p_1. For each $i = 1, \ldots, 4$, set $s_i' := s_i + (y, -2y)$, where y is the height of s_i. If s_2' is on p_3, then we define k as the number of steps of p_3 between s_1 and s_2'. Then define conditions $(\mathbf{F_2^{13}\text{-}a})$ and $(\mathbf{F_2^{13}\text{-}b})$ for $\mathbf{p} \in P_1^{23}(C_n; \mu, \lambda)$ as follows:

$(\mathbf{F_2^{13}\text{-}a})$ $\mathbf{p}$ satisfies all of the following conditions:

- s_1' is on p_1.
- s_2' is on p_3 and k is odd.

$(\mathbf{F_2^{13}\text{-}b})$ $\mathbf{p}$ satisfies all of the following conditions:

- s_1' is not on p_1.
- s_2' is on p_3 and k is odd.
- s_3' is on p_1.
- s_4' is on p_2.

We also define conditions $(\mathbf{F_2^{23}\text{-}a})$ and $(\mathbf{F_2^{23}\text{-}b})$ for $\mathbf{p} \in P_1^{12}(C_n; \mu, \lambda)$ as follows:

$(\mathbf{F_2^{23}\text{-}a})$ $\omega(\mathbf{p})$ satisfies the condition $(\mathbf{F_2^{13}\text{-}a})$.

$(\mathbf{F_2^{23}\text{-}b})$ $\omega(\mathbf{p})$ satisfies the condition $(\mathbf{F_2^{13}\text{-}b})$.

For example, $\mathbf{p} \in P_1^{23}(C_n; \mu, \lambda)$ in figure 14 satisfies the condition $(\mathbf{F_2^{13}\text{-}b})$. Note that $(\mathbf{F_2^{13}\text{-}a})$ and $(\mathbf{F_2^{13}\text{-}b})$ (resp. $(\mathbf{F_2^{23}\text{-}a})$ and $(\mathbf{F_2^{23}\text{-}b})$) are exclusive with each other.

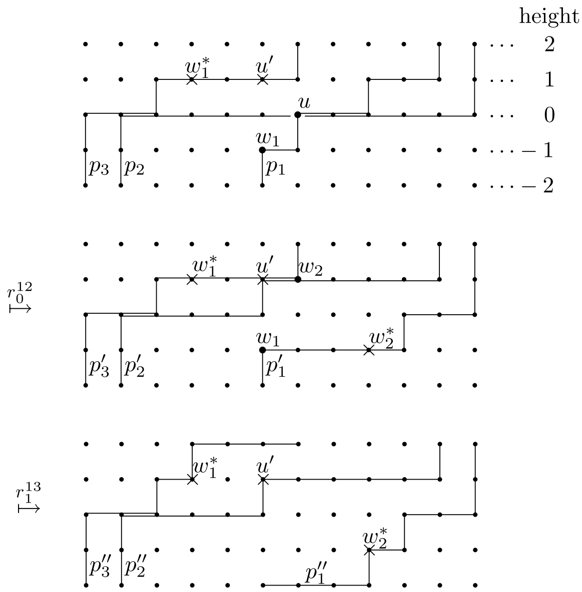


Figure 15. An example of $\mathbf{p} \in P_1^{12}(C_n; \mu, \lambda)$ and the map f_1^{12} of case (f_1^{12}-b).

We have

Lemma B.4.

(1) For $\mathbf{p} \in P_1(C_n; \mu, \lambda)$,

(1) $\mathbf{p} \in \mathrm{Im}\, f_2^{13}$ *if and only if either* $(\mathbf{F_2^{13}}\text{-}\mathbf{a})$ *or* $(\mathbf{F_2^{13}}\text{-}\mathbf{b})$ *is satisfied.*

(2) $\mathbf{p} \in \mathrm{Im}\, f_2^{23}$ *if and only if either* $(\mathbf{F_2^{23}}\text{-}\mathbf{a})$ *or* $(\mathbf{F_2^{23}}\text{-}\mathbf{b})$ *is satisfied.*

(2) f_2^{13} *and* f_2^{23} *are weight-preserving, sign-inverting injections.*

Proof. We prove it for f_2^{13}. We can check that $\mathbf{p}$ in the image of $(f_2^{13}\text{-a})$ satisfy $(\mathbf{F_2^{13}}\text{-}\mathbf{a})$. Conversely, one can invert the procedure of $(f_2^{13}\text{-a})$ for any $\mathbf{p} \in P_1^{23}(C_n; \mu, \lambda)$ that satisfies $(\mathbf{F_2^{13}}\text{-}\mathbf{a})$. The same holds when $(f_2^{13}\text{-a})$ (resp. $(\mathbf{F_2^{13}}\text{-}\mathbf{a})$) are replaced with $(f_2^{13}\text{-b})$ (resp. $(\mathbf{F_2^{13}}\text{-}\mathbf{b})$) (see figure 14 for example). □

The maps f_1^{12} and f_1^{23}. We define a map

$$f_1^{12} : P_1^{12}(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda),$$

as follows, using the weight-preserving maps r_y^{ij} defined in (B.2) (roughly speaking, f_1^{12} resolves the transposed pair (p_1, p_2) of $\mathbf{p} = (p_1, p_2, p_3)$ without producing any ordinarily intersecting paths): for any $\mathbf{p} = (p_1, p_2, p_3) \in P_1^{12}(C_n; \mu, \lambda)$ (see figure 15), let w_1 be the leftmost point on p_1 at height -1 and $w_1^* = w_1 + (-2, 2)$. Let u be the leftmost intersecting point of p_1 and p_2 and $u' := u + (-1, 1)$. Set $\mathbf{p}' := (p_1', p_2', p_3') = r_0^{12}(\mathbf{p})$, which is well defined. Then

Case $(f_1^{12}\text{-a})$. If u' is not on any p_i, set $f_1^{12}(\mathbf{p}) = \mathbf{p}'$, which is in $P_0(C_n; \mu, \lambda)$. (Otherwise, u' is on p_3 and (p_2', p_3') is ordinarily intersecting.)

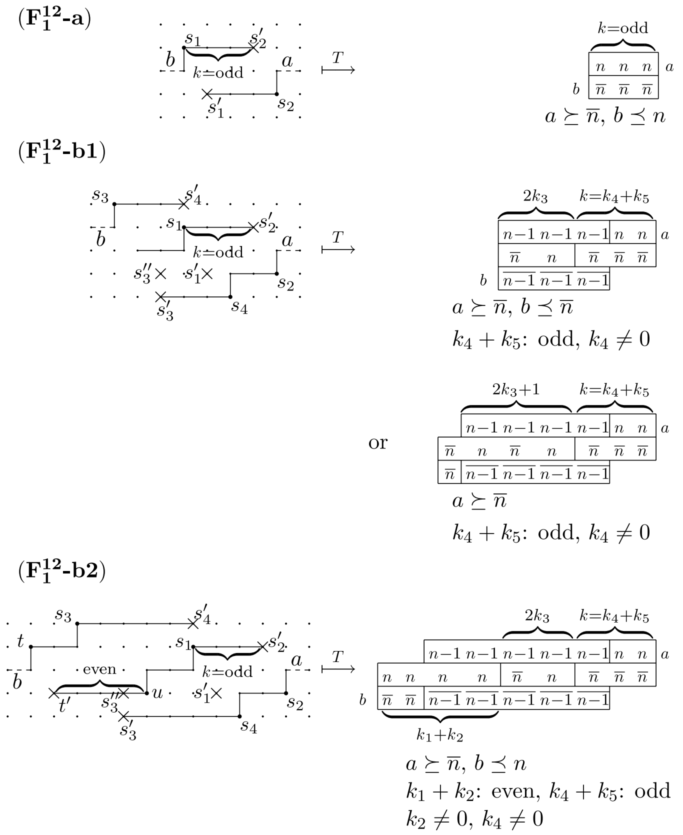


Figure 16. Examples of $\mathbf{p} \in P_0(C_n; \mu, \lambda)$ which satisfy one of the conditions of $\mathbf{p} \in \mathrm{Im}\, f_1^{12}$ in lemma B.5 (1) and the corresponding subtableau in $T(\mathbf{p})$. If the (E) step for a (resp. b) exists, then a (resp. b) satisfies the condition as above.

Case $\left(f_1^{12}\text{-b}\right)$. If u' is on p_3 and w_1^* is on p_3, then (p_1', p_3') satisfies $(\mathbf{R}_1)$. Set $f_1^{12}(\mathbf{p}) = r_1^{13}(\mathbf{p}')$, which is in $P_0(C_n; \mu, \lambda)$. (If w_1^* is not on p_3, then $r_1^{13}(\mathbf{p}')$ is not defined.)

Case $\left(f_1^{12}\text{-c}\right)$. Otherwise, (p_2, p_3) satisfies $(\mathbf{R}_0)$. If we set $\mathbf{p}'' = r_0^{23}(\mathbf{p}') = (p_1'', p_2'', p_3'')$, then (p_1'', p_2'') is ordinarily intersecting. For (p_1'', p_3'') satisfies $(\mathbf{R}_1)$, we set $f_1^{12}(\mathbf{p}) = r_1^{13}(\mathbf{p}'')$, which is in $P_0(C_n; \mu, \lambda)$.

We also define

$$f_1^{23} : P_1^{23}(C_n; \mu, \lambda) \to P_0(C_n; \mu, \lambda)$$

as $f_1^{23} := \omega \circ f_1^{12} \circ \omega$, where ω is the map defined in (B.3). Then f_1^{23} resolves the transposed pair (p_2, p_3) of $\mathbf{p} = (p_1, p_2, p_3)$.

Next, we give the conditions to describe the image of f_1^{12} and f_1^{23}. For any $\mathbf{p} \in P_0(C_n; \mu, \lambda)$, let s_i and s_i' $(i = 1, \ldots, 4)$ be the points as in the conditions $\left(\mathbf{F_1^{12}\text{-a}}\right)$ and $\left(\mathbf{F_1^{12}\text{-b}}\right)$, with the roles of p_2 and p_3 interchanged. Namely, (see figure 16) let s_1 be the leftmost point of height 1 on p_2, s_2 be the rightmost point of height -1 on p_1, s_3 be the leftmost point of height 2 on p_3, and s_4 be the rightmost point of height -2 on p_1, and set $s_i' := s_i + (y, -2y)$, where y is the height of s_i. If s_2' is on p_2, then let k be the number of steps of p_2 between s_1 and s_2. Then define conditions $\left(\mathbf{F_1^{12}\text{-a}}\right)$ and $\left(\mathbf{F_1^{12}\text{-b}}\right)$ for $\mathbf{p} \in P_0(C_n; \mu, \lambda)$

similar to the conditions $\left(\mathbf{F}_2^{13}\textbf{-a}\right)$ and $\left(\mathbf{F}_2^{13}\textbf{-b}\right)$, with the roles of p_2 and p_3 in the conditions interchanged. Namely,

$\left(\mathbf{F}_1^{12}\textbf{-a}\right)$ $\mathbf{p}$ satisfies all of the following conditions:

- s_1' is on p_1.
- s_2' is on p_2 and k is odd.

$\left(\mathbf{F}_1^{12}\textbf{-b}\right)$ $\mathbf{p}$ satisfies all of the following conditions:

- s_1' is not on p_1.
- s_2' is on p_2 and k is odd.
- s_3' is on p_1.
- s_4' is on p_3.

We also define conditions $\left(\mathbf{F}_1^{23}\textbf{-a}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-b}\right)$ for $\mathbf{p} \in P_0(C_n; \mu, \lambda)$ as follows:

$\left(\mathbf{F}_1^{23}\textbf{-a}\right)$ $\omega(\mathbf{p})$ satisfies the condition $\left(\mathbf{F}_1^{12}\textbf{-a}\right)$.
$\left(\mathbf{F}_1^{23}\textbf{-b}\right)$ $\omega(\mathbf{p})$ satisfies the condition $\left(\mathbf{F}_1^{12}\textbf{-b}\right)$.

Note that $\left(\mathbf{F}_1^{12}\textbf{-a}\right)$ and $\left(\mathbf{F}_1^{12}\textbf{-b}\right)$ (resp. $\left(\mathbf{F}_1^{23}\textbf{-a}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-b}\right)$) are exclusive with each other.

For any $\mathbf{p} = (p_1, p_2, p_3) \in P_0(C_n; \mu, \lambda)$, let s_3 be the leftmost point of height 2 on p_3 (as we defined in $\left(\mathbf{F}_1^{23}\textbf{-a}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-b}\right)$) and $s_3'' := s_3 + (2, -3)$. Let t be the leftmost point of height 1 on p_3 and $t' := t + (1, -2)$. Let u be the rightmost point of height -1 on p_2. We define conditions $\left(\mathbf{F}_1^{12}\textbf{-b1}\right)$, $\left(\mathbf{F}_1^{12}\textbf{-b2}\right)$, $\left(\mathbf{F}_1^{23}\textbf{-b1}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-b2}\right)$ as follows (see figure 16):

$\left(\mathbf{F}_1^{12}\textbf{-b1}\right)$ $\mathbf{p}$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-b}\right)$ and s_3'' is not on p_2.
$\left(\mathbf{F}_1^{12}\textbf{-b2}\right)$ $\mathbf{p}$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-b}\right)$, s_3'' is on p_2, t' is on p_2, and the number of $\left(\mathbf{F}_1^{12}\textbf{-b2}\right)$ the steps from t' to u is even.
$\left(\mathbf{F}_1^{23}\textbf{-b1}\right)$ $\omega(\mathbf{p})$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-b1}\right)$.
$\left(\mathbf{F}_1^{23}\textbf{-b2}\right)$ $\omega(\mathbf{p})$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-b2}\right)$.

Then we have

Lemma B.5.

(1) Let $\mathbf{p} \in P_0(C_n; \mu, \lambda)$. Then, $\mathbf{p} \in \operatorname{Im} f_1^{12}$ if and only if one of the conditions $\left(\mathbf{F}_1^{12}\textbf{-a}\right)$, $\left(\mathbf{F}_1^{12}\textbf{-b1}\right)$ and $\left(\mathbf{F}_1^{12}\textbf{-b2}\right)$ is satisfied. Similarly, $\mathbf{p} \in \operatorname{Im} f_1^{23}$ if and only if one of the conditions $\left(\mathbf{F}_1^{23}\textbf{-a}\right)$, $\left(\mathbf{F}_1^{23}\textbf{-b1}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-b2}\right)$ is satisfied.
(2) f_1^{12} and f_1^{23} are weight-preserving sign-inverting injections.

The proof of lemma B.5 is similar to that of lemma B.4.

Finally, we give two lemmas which are used in section 5.3.

Lemma B.6.

(1) $\operatorname{Im} f_1^{12} \cap \operatorname{Im} f_1^{23} = \operatorname{Im}\left(f_1^{23} \circ f_2^{13}\right)$.
(2) $\operatorname{Im}\left(f_1^{12} \circ f_2^{23}\right) = \operatorname{Im}\left(f_1^{23} \circ f_2^{13}\right)$.

Proof. Using the conditions in lemma B.5 (1), we can check that $\mathbf{p} \in \operatorname{Im} f_1^{12} \cap \operatorname{Im} f_1^{23}$ if and only if $\mathbf{p}$ satisfies one of the following:

(a) $\mathbf{p}$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-a}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-a}\right)$.
(b) $\mathbf{p}$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-a}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-b}\right)$ $\left(\Leftrightarrow \left(\mathbf{F}_1^{12}\textbf{-a}\right) \text{ and } \left(\mathbf{F}_1^{23}\textbf{-b1}\right)\right)$.
(c) $\mathbf{p}$ satisfies $\left(\mathbf{F}_1^{12}\textbf{-b}\right)$ and $\left(\mathbf{F}_1^{23}\textbf{-a}\right)$ $\left(\Leftrightarrow \left(\mathbf{F}_1^{12}\textbf{-b1}\right) \text{ and } \left(\mathbf{F}_1^{23}\textbf{-a}\right)\right)$.

On the other hand, $f_1^{23} \circ f_2^{13}$ is given as one of the following cases, by the conditions of $\mathbf{p} \in \operatorname{Im} f_2^{13}$:

(1) f_2^{13} as in case $\left(f_2^{13}\text{-a}\right)$ and f_1^{23} as in case $\left(f_1^{23}\text{-a}\right)$.
(2) f_2^{13} as in case $\left(f_2^{13}\text{-a}\right)$ and f_1^{23} as in case $\left(f_1^{23}\text{-b}\right)$.
(3) f_2^{13} as in case $\left(f_2^{13}\text{-b}\right)$ and f_1^{23} as in case $\left(f_1^{23}\text{-a}\right)$.

As in the proof of lemma B.4, all $\mathbf{p} \in \operatorname{Im} f_2^{13}$ of case $\left(f_2^{13}\text{-a}\right)$ $\left(\text{resp. case}\left(f_2^{13}\text{-b}\right)\right)$ satisfy $\left(\mathbf{F_2^{13}\text{-a}}\right)$ $\left(\text{resp. } \left(\mathbf{F_2^{13}\text{-b}}\right)\right)$. Since the conditions $\left(\mathbf{F_2^{13}\text{-a}}\right)$ and $\left(\mathbf{F_2^{13}\text{-b}}\right)$ of $\mathbf{p} \in \operatorname{Im} f_2^{13}$ turn out to be the conditions $\left(\mathbf{F_1^{12}\text{-a}}\right)$ and $\left(\mathbf{F_1^{12}\text{-b}}\right)$ respectively after $\mathbf{p}$ is sent by f_1^{23}, all $\mathbf{p} \in \operatorname{Im} f_1^{23} \circ f_2^{13}$ of case (1) satisfy (a), while that of (2) satisfy (b) and that of (3) satisfy (c). Thus, we obtain $\operatorname{Im}\left(f_1^{23} \circ f_2^{13}\right) \subset \operatorname{Im} f_1^{12} \cap \operatorname{Im} f_1^{23}$. Conversely, $f_1^{12} \cap \operatorname{Im} f_1^{23} \subset \operatorname{Im}\left(f_1^{23} \circ f_2^{13}\right)$ is obvious, and therefore, (1) is proved. The condition of $\operatorname{Im}\left(f_1^{12} \circ f_2^{23}\right)$ is given similarly, and we obtain (2). □

Lemma B.7. *For* $\mathbf{p} \in \tilde{P}(C_n; \mu, \lambda)$, $\mathbf{p} \in \operatorname{Im} f_1^{12} \cup \operatorname{Im} f_1^{23}$ *if and only if* $T(\mathbf{p}) \notin \operatorname{Tab}(C_n, \lambda/\mu)$, *where* $\operatorname{Tab}(C_n, \lambda/\mu)$ *is the set of tableaux defined in section 5.3.*

Proof. If $\mathbf{p} \in P_0(C_n; \mu, \lambda)$ satisfies one of the conditions of $\mathbf{p} \in \operatorname{Im} f_1^{12}$ or that of $\mathbf{p} \in \operatorname{Im} f_1^{23}$ in lemma B.5 (1), then $T(\mathbf{p})$ does not satisfy either the extra rule (**E-2R**) or the extra rule (**E-3R**) (see figure 16).

Conversely, let $\mathbf{p} \in \tilde{P}(C_n; \mu, \lambda)$ and $T(\mathbf{p}) \notin \operatorname{Tab}(C_n, \lambda/\mu)$. By lemma B.3 (2), there does not exist any $\mathbf{p} \in \operatorname{Im} g$ such that $T(\mathbf{p})$ contains one of subtableaux (5.6), (5.7), (5.8) and (5.9), and therefore, $T(\mathbf{p}) \notin \operatorname{Tab}(C_n, \lambda/\mu)$ implies that $\mathbf{p} \in P_0(C_n; \mu, \lambda)$, by lemma B.3 (1). By assumption, $T(\mathbf{p})$ contains a subtableau T' described as in (5.6), (5.7), (5.8) or (5.9) which does not satisfy the extra rule (**E-2R**) or (**E-3R**). We can check that $\mathbf{p}$ satisfies one of the conditions in lemma B.5 (1) for all such T'. Namely (see figure 16),

(1) If T' is subtableau (5.6) prohibited by the extra rule (**E-2R**), then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{12}\text{-a}}\right)$ or $\left(\mathbf{F_1^{23}\text{-a}}\right)$.
(2) If T' is subtableau (5.7) prohibited by the extra rule (**E-3R**) and
   (a) If $k_4 + k_5$ is odd, $k_4 \neq 0$ and $k_2 = 0$, then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{12}\text{-b1}}\right)$.
   (b) If $k_4 + k_5$ is odd, $k_4 \neq 0$ and $k_2 \neq 0$, then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{12}\text{-b2}}\right)$.
   (c) If $k_4 + k_5$ is odd, $k_4 = 0$ and $k_2 \neq 0$, then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{12}\text{-a}}\right)$.
   (d) If $k_1 + k_2$ is odd, $k_2 \neq 0$ and $k_4 = 0$, then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{23}\text{-b1}}\right)$.
   (e) If $k_1 + k_2$ is odd, $k_2 \neq 0$ and $k_4 \neq 0$, then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{23}\text{-b2}}\right)$.
   (f) If $k_1 + k_2$ is odd, $k_2 \neq 0$ and $k_4 = 0$, then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{23}\text{-a}}\right)$.
(3) If T' is subtableau (5.8) (resp. (5.9)) prohibited by the extra rule (**E-3R**), then $\mathbf{p}$ satisfies $\left(\mathbf{F_1^{12}\text{-b1}}\right)$ $\left(\text{resp. } \left(\mathbf{F_1^{23}\text{-b1}}\right)\right)$. □

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